We claim that with $n \ge m \ge 0$

$$2^{n-m}3^m = (-1)^n \sum_{k=0}^{n} (-1)^k {m+3k \choose k} {n+2k \choose n-k}.$$

Start by observing that the second binomial coefficient enforces the upper range so that with the usual integrals we get

$$\frac{(-1)^n}{2\pi i} \int_{|w|=\gamma} \frac{(1+w)^m}{w} \frac{1}{2\pi i} \int_{|z|=\varepsilon} \frac{(1+z)^n}{z^{n+1}} \\ \times \sum_{k\ge 0} (-1)^k z^k (1+z)^{2k} (1+w)^{3k} w^{-k} \; dz \; dw.$$

For the sum to converge we require $|z(1+z)^2| < |w/(1+w)^3|$. With $\varepsilon < 1/2$ we have $|z(1+z)^2| < 9\varepsilon/4$ and with $\gamma < 1/2$ we have $|w/(1+w)^3| > 8\gamma/27$ thus we require $\varepsilon \le 32\gamma/243$. We may take $\gamma = 1/3^Q$ and $\varepsilon = 32/3^{Q+5}$ with Q large. Continuing,

$$\frac{(-1)^n}{2\pi i} \int_{|w|=\gamma} \frac{(1+w)^m}{w} \frac{1}{2\pi i} \int_{|z|=\varepsilon} \frac{(1+z)^n}{z^{n+1}} \frac{1}{1+z(1+z)^2(1+w)^3/w} \; dz \; dw \\ = \frac{(-1)^n}{2\pi i} \int_{|w|=\gamma} (1+w)^m \frac{1}{2\pi i} \int_{|z|=\varepsilon} \frac{(1+z)^n}{z^{n+1}} \frac{1}{w+z(1+z)^2(1+w)^3} \; dz \; dw.$$

Now put $z = v/(1+w)$ so that $dz = dv/(1+w)$ to get

$$\frac{(-1)^n}{2\pi i} \int_{|w|=\gamma} (1+w)^m \frac{1}{2\pi i} \int_{|v|=\rho(w)} \frac{(1+v/(1+w))^n}{v^{n+1}} (1+w)^{n+1} \\ \times \frac{1}{w+v(1+v/(1+w))^2(1+w)^3/(1+w)} \frac{1}{1+w} \; dv \; dw \\ = \frac{(-1)^n}{2\pi i} \int_{|w|=\gamma} (1+w)^m \frac{1}{2\pi i} \int_{|v|=\rho(w)} \frac{(1+w+v)^n}{v^{n+1}} \\ \times \frac{1}{w+v(1+w+v)^2} \; dv \; dw.$$

Here $|v| = \rho(w)$ is the image of scaling $|z| = \varepsilon$ by $1+w$ which is a circle of radius $\varepsilon|1+w|$ and is therefore contained in an annulus with inner radius $\varepsilon(1-\gamma)$ and outer radius $\varepsilon(1+\gamma)$. Factorizing,

$$\frac{(-1)^n}{2\pi i} \int_{|w|=\gamma} (1+w)^m \frac{1}{2\pi i} \int_{|v|=\rho(w)} \frac{(1+w+v)^n}{v^{n+1}} \\ \times \frac{1}{(v+w)(vw+(1+v)^2)} \; dv \; dw.$$

We see that with $\gamma \ll 1$ the circle $|v| = \rho(w)$ approximates the contour $|v| = \varepsilon$. The poles in v other than $v = 0$ are outside this contour, with the first one requiring that $|v| = \gamma$ whereas $|v|$ approximates ε and the other two becoming arbitrarily close to -1 as Q grows large. Therefore we are justified deforming the contour for v to the circle $|v| = \varepsilon$ independent of w since we do not pick up or lose any poles. Switching integrals with Fubini yields

$$\frac{(-1)^n}{2\pi i} \int_{|v|=\varepsilon} \frac{1}{v^{n+1}} \frac{1}{2\pi i} \int_{|w|=\gamma} (1+w)^m (1+w+v)^n \\ \times \frac{1}{(v+w)(vw+(1+v)^2)} \; dw \; dv.$$

With the inner integral in w we clearly have the pole at $w=-v$ inside the contour by construction of ε and γ. On the other hand the pole at $w=-(1+v)^2/v$ goes to infinity as Q is large and is therefore outside the contour. Therefore the inner integral is the residue at $w=-v$ which yields for the remaining outer integral

$$\frac{(-1)^n}{2\pi i} \int_{|v|=\varepsilon} \frac{1}{v^{n+1}} (1-v)^m \frac{1}{1+2v} \; dv.$$

This is

$$(-1)^n \sum_{q=0}^{n} (-1)^q \binom{m}{q} (-1)^{n-q} 2^{n-q} = 2^n \sum_{q=0}^{n} \binom{m}{q} 2^{-q}.$$

Recall that we said $n \ge m$ which yields at last

$$2^n \left(1+\frac{1}{2}\right)^m = 2^{n-m} 3^m.$$

We have the claim and may conclude. We may use the sum that appeared last as a natural way to extend a closed form to the case when $m > n$.

This identity was found by a computer search which pointed to [OEIS A036561](https://oeis.org/A036561), Nicomachus triangle.