Factorisations of some Riordan arrays as infinite products

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We find two factorisations of an element of the Bell subgroup of the Riordan group as an infinite product of arrays and also two factorisations of an element of the derivative subgroup of the exponential Riordan group as an infinite product of arrays.

1. The forward arrow operator $\to$

Let $M = (M(n,k))_{n,k\geq 0}$ be an infinite lower triangular array and let I_k, $k = 0, 1, 2, \ldots$, denote the square k x k identity matrix (all our arrays have row and column indices starting at 0). Define $M(k)$ as the infinite lower triangular block array

$$M(k) = \begin{pmatrix} I_k & 0 \\ \\ 0^T & M \end{pmatrix}$$

so, in particular, $M(0) = M$. Define $\overrightarrow{M}$ as the infinite matrix product

$$\overrightarrow{M} = M(0)M(1)M(2)\cdots . \tag{1}$$

Clearly, $\overrightarrow{M}$ is well-defined.

Example 1.1 *Let U be the lower triangular array with all entries on and below the main diagonal equal to 1. Let P denote Pascal's triangle A007318. Then $\overrightarrow{U} = P$.*

$$\begin{pmatrix} 1 & 0 & 0 & 0 & \ldots \\ 1 & 1 & 0 & 0 & \ldots \\ 1 & 1 & 1 & 0 & \ldots \\ 1 & 1 & 1 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & \ldots \\ 0 & 1 & 0 & 0 & \ldots \\ 0 & 1 & 1 & 0 & \ldots \\ 0 & 1 & 1 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & \ldots \\ 0 & 1 & 0 & 0 & \ldots \\ 0 & 0 & 1 & 0 & \ldots \\ 0 & 0 & 1 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \cdots = \begin{pmatrix} 1 & 0 & 0 & 0 & \ldots \\ 1 & 1 & 0 & 0 & \ldots \\ 1 & 2 & 1 & 0 & \ldots \\ 1 & 3 & 3 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

This is a well-known result [1, Theorem 1].

Example 1.2. *Let $S2$ denote the triangle of Stirling numbers of the second kind A008277 (but with different row and column indexing from that used in the OEIS). Then $\overrightarrow{P} = S2$* [3, Theorem 2.2].

$$\begin{pmatrix} 1 & 0 & 0 & 0 & \ldots \\ 1 & 1 & 0 & 0 & \ldots \\ 1 & 2 & 1 & 0 & \ldots \\ 1 & 3 & 3 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & \ldots \\ 0 & 1 & 0 & 0 & \ldots \\ 0 & 1 & 1 & 0 & \ldots \\ 0 & 1 & 2 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & \ldots \\ 0 & 1 & 0 & 0 & \ldots \\ 0 & 0 & 1 & 0 & \ldots \\ 0 & 0 & 1 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \cdots = \begin{pmatrix} 1 & 0 & 0 & 0 & \ldots \\ 1 & 1 & 0 & 0 & \ldots \\ 1 & 3 & 1 & 0 & \ldots \\ 1 & 7 & 6 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

The Hockey-Stick identity

From the definition (1) of the forward arrow operator we have

$$\overrightarrow{M(1)} = M(1)M(2)M(3)\cdots = \begin{pmatrix} 1 & 0 \\ & \\ 0^T & \overrightarrow{M} \end{pmatrix}.$$

Hence, from (1),

$$\overrightarrow{M} = M \begin{pmatrix} 1 & 0 \\ & \\ 0^T & \overrightarrow{M} \end{pmatrix}. \tag{2}$$

An immediate consequence of (2) is that columns 0 of M and $\overrightarrow{M}$ are equal. Equating entries in position $(n+1, k+1)$ on both sides of (2) yields a 'vertical' recurrence equation for the entries of $\overrightarrow{M}$:

$$\overrightarrow{M}(n+1, k+1) = \sum_{i=k}^{n} M(n+1, i+1)\overrightarrow{M}(i, k) \tag{3}$$

with the boundary conditions $\overrightarrow{M(n,0)} = M(n,0)$ for $n = 0, 1, 2, ...$ This recurrence is called the hockey-stick identity.

For instance, for Example 1.2, the hockey-stick identity reads

$$\text{Stirling2}(n+1, k) = \sum_{i=k-1}^{n} \binom{n}{i} \text{Stirling2}(i, k-1),$$

a well-known recurrence for the Stirling numbers of the second kind.

Remark 1. If M is invertible then we can rewrite (2) as

$$M^{-1}\overrightarrow{M} = \begin{pmatrix} 1 & 0 \\ & \\ 0^T & \overrightarrow{M} \end{pmatrix}$$

and obtain a second recurrence for the elements of $\overrightarrow{M}$:

$$\overrightarrow{M}(n, k) = \sum_{i=0}^{n+1} M^{-1}(n+1, i)\overrightarrow{M}(i, k+1). \tag{4}$$

In the case of Example 1.2, we obtain a less well-known recurrence for the Stirling numbers of the second kind:

$$\text{Stirling2}(n, k) = \sum_{i=0}^{n} (-1)^{n+i} \binom{n}{i} \text{Stirling2}(i+1, k+1).$$

2. The backward arrow operator $\leftarrow$

We also consider infinite matrix products running from right to left. Given a lower triangular array M, we define the array $\overleftarrow{M}$ as the infinite matrix product

$$\overleftarrow{M} = \cdots M(2)M(1)M(0)\,. \tag{5}$$

The forward and backward arrow operators are related via matrix inversion (assuming M is invertible) and matrix transposition as follows:

$$\overleftarrow{M} = \left(\overrightarrow{M^{-1}}\right)^{-1} = \left(\overrightarrow{M^{T}}\right)^{T}. \tag{6}$$

Clearly, from (5), we have

$$\overleftarrow{M} = \begin{pmatrix} 1 & 0 \\ & \\ 0^T & \overleftarrow{M} \end{pmatrix} M \tag{7}$$

leading to a 'horizontal' recurrence equation for the entries of $\overleftarrow{M}$:

Row $k = 0 : \overleftarrow{M}(0,0) = M(0,0)$ and for $n \geq k \geq 0$,

$$\overleftarrow{M}(n+1,k) = \sum_{i=0}^{n} M(i+1,k)\overleftarrow{M}(n,i). \tag{8}$$

Remark 2. If M is invertible then further relations between the entries of $\overleftarrow{M}$ can be found by rewriting (7) as

$$\overleftarrow{M}M^{-1} = \begin{pmatrix} 1 & 0 \\ & \\ 0^T & \overleftarrow{M} \end{pmatrix}.$$

For example, when $k \geq 1$, we have

$$\overleftarrow{M}(n,k) = \sum_{i=k+1}^{n+1} M^{-1}(i,k+1)\overleftarrow{M}(n+1,i). \tag{9}$$

Example 2.1. $\overleftarrow{U} = C$, *where C denotes the Catalan triangle A033184.*

$$\cdots \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & \cdots \\ 0 & 1 & 1 & 0 & \cdots \\ 0 & 1 & 1 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & \cdots \\ 1 & 1 & 1 & 0 & \cdots \\ 1 & 1 & 1 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & \cdots \\ 2 & 2 & 1 & 0 & \cdots \\ 5 & 5 & 3 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

For this example, the recurrences (8) and (9) give

$$C(n+1,k) = \sum_{i=k-1}^{n} C(n,i) \quad \text{for } k \geq 1$$

and

$$C(n+1,k) = C(n,k-1) + C(n+1,k+1) \quad \text{for } k \geq 1.$$

Example 2.1 is a particular case of Theorem 1 (ii) proved below. Another case of Theorem 1 (ii) is the following known result.

Example 2.2. *Let $S1$ denote the triangle of unsigned Stirling numbers of the first kind A130534.. Then $\overleftarrow{P} = S1$.*

$$\cdots \begin{pmatrix} 1 & 0 & 0 & 0 & \ldots \\ 0 & 1 & 0 & 0 & \ldots \\ 0 & 0 & 1 & 0 & \ldots \\ 0 & 0 & 1 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & \ldots \\ 0 & 1 & 0 & 0 & \ldots \\ 0 & 1 & 1 & 0 & \ldots \\ 0 & 1 & 2 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & \ldots \\ 1 & 1 & 0 & 0 & \ldots \\ 1 & 2 & 1 & 0 & \ldots \\ 1 & 3 & 3 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & \ldots \\ 1 & 1 & 0 & 0 & \ldots \\ 2 & 3 & 1 & 0 & \ldots \\ 6 & 11 & 6 & 1 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

A common feature shared by the above examples is that the arrays U, P, $S1$, $S2$ and C are examples of Riordan arrays, either in the Riordan group or the exponential Riordan group. This suggests we look at the action of the forward and backward arrow operators on Riordan arrays.

3. Riordan arrays

We recall some basic facts about Riordan arrays. Riordan arrays are a special type of infinite lower triangular matrices defined by two generating functions

$$f(x) = f_0 + f_1x + f_2x^2 + \cdots$$
$$g(x) = g_1x + g_2x^2 + g_3x^3 + \cdots$$

with $f_0 \neq 0$ and $g_1 \neq 0$. The *Riordan array* associated with this pair of series, denoted by $(f(x), g(x))$, is defined as the infinite lower triangular array whose k-th column is formed from the coefficients of the power series $f(x)g(x)^k$, $k = 0, 1, 2, ...$

The *Riordan group* is the set of all Riordan arrays with the group operation being matrix multiplication. The group law is given by

$$(f(x), g(x)) * (h(x), l(x)) = (f(x)h(g(x)), l(g(x))). \tag{10}$$

The identity element of the Riordan group is $(1, x)$. The inverse of the array $(f(x), g(x))$ is the array $\left(\frac{1}{f(g^{<-1>}(x))}, g^{<-1>}(x)\right)$, where $g^{<-1>}(x)$ denotes the compositional inverse of $g(x)$, that is, $g(g^{<-1>}(x)) = g^{<-1>}(g(x)) = x$.

The two sets of Riordan arrays of the form $(f(x), x)$ and $(f(x), xf(x))$ are easily seen to be subgroups of the Riordan group, known respectively as the the *Appell subgroup* and the *Bell subgroup*.

For the arrays considered in the above examples, $U = \left(\frac{1}{1-x}, x\right)$ belongs to the Appell subgroup of the Riordan group. Both Pascal's triangle $P = \left(\frac{1}{1-x}, \frac{x}{1-x}\right)$ and the Catalan triangle $C = (c(x), xc(x))$, where $c(x) = \frac{1-\sqrt{1-4x}}{2x}$ is the generating function of the Catalan numbers A000108, belong to the Bell subgroup of the Riordan group.

Exponential Riordan arrays are a special type of infinite lower triangular matrices defined by two exponential generating functions (e.g.f's)

$$f(x) = f_0 + f_1 x/1! + f_2 x^2/2! + \cdots$$

$$g(x) = g_1 x/1! + g_2 x^2/2! + g_3 x^3/3! + \cdots$$

with $f_0 \neq 0$ and $g_1 \neq 0$. The *exponential Riordan array* associated with this pair of series, denoted by $[f(x), g(x)]$, is defined as the infinite lower triangular array whose k-th column has the e.g.f. $\frac{1}{k!} f(x) g(x)^k$, $k = 0, 1, 2, \ldots$. The *exponential Riordan group* is the set of all Riordan arrays with the group operation being matrix multiplication. The group law is the same as (10).

The set of exponential Riordan arrays of the form $\left[\frac{dg(x)}{dx}, g(x)\right]$ form a subgroup of the exponential Riordan group called the *derivative subgroup*. The triangles $S1$ and $S2$ of Stirling numbers of the first and second kinds both lie in the derivative subgroup of the exponential Riordan group.

$$S1 = \left[\frac{1}{1-x}, \log\left(\frac{1}{1-x}\right)\right], \qquad S2 = \left[e^x, e^x - 1\right].$$

4. The action of the forward and backward arrow operators on the Appell subgroup of the Riordan group

The examples $\overrightarrow{U} = P$ and $\overleftarrow{U} = C$ given above are particular cases of our first result below showing that both the forward and backward arrow operators map the Appell subgroup of the Riordan group onto the Bell subgroup of the Riordan group. In what follows, the superscript <-1> indicates series reversion while the superscript -1 indicates matrix inversion.

Theorem 1. *Let $M = (f(x), x)$ be a Riordan array in the Appell subgroup of the Riordan group. Then*
(i)

$$\overrightarrow{M} = (f(x), xf(x))$$

belongs to the Bell subgroup of the Riordan group.

(ii) *The array*

$$\overleftarrow{M} = (F(x), xF(x)),$$

where

$$F(x) = \frac{1}{x}\left(\frac{x}{f(x)}\right)^{<-1>}.$$

$\overleftarrow{M}$ also belongs to the Bell subgroup of the Riordan group.

Proof.

(i) The Riordan array $(f(x), xf(x))$ factorises in the Riordan group as

$$(f(x), xf(x)) \quad = \quad (f(x), x)\,(1, xf(x))\,. \tag{11}$$

The k-th column of the Riordan array $(1, xf(x))$ has the o.g.f. $(xf(x))^k$. If we write the array $(1, xf(x))$ in block diagonal form as $\begin{bmatrix} 1 & 0 \\ 0 & X \end{bmatrix}$ then, for $k \geq 0$, the k-th column of the array X has the o.g.f. $\frac{1}{x}(xf(x))^{k+1} = x^k f(x)^{k+1}$. Thus X is the Riordan array $(f(x), xf(x))$ and by (11) we have the factorisation

$$\begin{aligned} (f(x), xf(x)) &= (f(x), x)\,(1, xf(x)) \\ &= (f(x), x)\begin{pmatrix} 1 & | & 0 \\ - & | & - \\ 0^T & | & (f(x), xf(x)) \end{pmatrix}. \end{aligned} \tag{12}$$

Iterating (12) yields

$$\begin{aligned} (f(x), xf(x)) &= (f(x), x)\begin{pmatrix} 1 & | & 0 \\ - & | & - \\ 0^T & | & \overrightarrow{(f(x), x)} \end{pmatrix} \\ &= \overrightarrow{(f(x), x)} \end{aligned}$$

by (2).

(ii) By (6), the inverse Riordan array

$$\begin{aligned}\left(\overleftarrow{(f(x),x)}\right)^{-1} &= \overrightarrow{\left((f(x),x)^{-1}\right)} \\ &= \overrightarrow{\left(\frac{1}{f(x)}, x\right)} \\ &= \left(\frac{1}{f(x)}, \frac{x}{f(x)}\right)\end{aligned}$$

by part (i).

Hence,

$$\begin{aligned}\overleftarrow{(f(x),x)} &= \left(\frac{1}{f(x)}, \frac{x}{f(x)}\right)^{-1} \\ &= \left(f\left(\left(\frac{x}{f(x)}\right)^{<-1>}\right), \left(\frac{x}{f(x)}\right)^{<-1>}\right) \\ &= \left(\frac{1}{x}\left(\frac{x}{f(x)}\right)^{<-1>}, \left(\frac{x}{f(x)}\right)^{<-1>}\right)\end{aligned}$$

where, in the final step, we used the fact that if $g(x) = (x/f(x))^{<-1>}$ then $g(x)/f(g(x)) = x$, and so $f(g(x)) = g(x)/x = 1/x(x/f(x))^{<-1>}$. ■

Example 4.1.

$$\overrightarrow{U^k} = \overrightarrow{\left(\frac{1}{1-kx}, x\right)} = \left(\frac{1}{1-kx}, \frac{x}{1-kx}\right) = P^k = (\overrightarrow{U})^k.$$

Example 4.2.

$$\begin{aligned}\overleftarrow{U^k} &= \overleftarrow{\left(\frac{1}{1-kx}, x\right)} = \left(\frac{1}{x}(x(1-kx))^{<-1>}, (x(1-kx))^{<-1>}\right) \\ &= (c(kx), xc(kx)),\end{aligned}$$

where $c(x) = \dfrac{1-\sqrt{1-4x}}{2x}$ is the generating function of the Catalan numbers.

5. The action of the forward and backward arrow operators on the derivative subgroup of the exponential Riordan group

Recall that Pascal's triangle P is also the exponential Riordan array $[e^x, x]$ belonging to the Appell subgroup of the exponential Riordan group. The examples $\overrightarrow{P} = S2 = [e^x, e^x - 1]$ and $\overleftarrow{P} = S1 = \left[\frac{1}{1-x}, \log\frac{1}{1-x}\right]$ given above are particular cases of our next theorem showing that both the forward and backward arrow operators map the Appell subgroup of the exponential Riordan group onto the derivative subgroup of the exponential Riordan group. The proofs are essentially the same as in Theorem 1, the main difference is that when working with ordinary generating functions division by x acts as a lowering operator on the monomial polynomials:

$$\frac{1}{x}x^n = x^{n-1};$$

the corresponding operator when working with exponential generating functions is differentiation:

$$\frac{d}{dx}\left(\frac{x^n}{n!}\right) = \frac{x^{n-1}}{(n-1)!}.$$

Theorem 2. *Let $M = [f(x), x]$ be an exponential Riordan array belonging to the exponential Appell group. Then*

(i)

$$\overrightarrow{M} = \left[f(x), \int_0^x f(t)\,dt\right]$$

belongs to the derivative subgroup of the exponential Riordan group.

(ii)

$$\overleftarrow{M} = \left[F(x), \int_0^x F(t)\,dt\right],$$

where

$$F(x) = \frac{d}{dx}\left(\left(\int_0^x \frac{dt}{f(t)}\right)^{<-1>}\right).$$

$\overleftarrow{M}$ also belongs to the derivative subgroup of the exponential Riordan group.

Proof.

(i) The exponential Riordan array $\left[f(x), \int_0^x f(t)\,dt\right]$ factorises as

$$\left[f(x), \int_0^x f(t)\,dt\right] \quad = \quad [f(x), x]\left[1, \int_0^x f(t)\,dt\right]. \qquad (13)$$

The k-th column in the exponential Riordan array $\left[1, \int_0^x f(t)\,dt\right]$ has the e.g.f. $\frac{1}{k!}\left(\int_0^x f(t)\,dt\right)^k$. If we write the array array $\left[1, \int_0^x f(t)\,dt\right]$ in block diagonal form as $\begin{bmatrix} 1 & 0 \\ 0^T & X \end{bmatrix}$ then, for $k \geq 0$, the k-th column in the array X has the e.g.f.

$$\frac{d}{dx}\left(\frac{1}{(k+1)!}\left(\int_0^x f(t)\,dt\right)^{k+1}\right) = \frac{1}{k!}\left(\int_0^x f(t)\,dt\right)^k f(x).$$

Thus X is the exponential Riordan array $\left[f(x), \int_0^x f(t)\,dt\right]$.

Hence, from (13),

$$\begin{aligned}\left[f(x), \int_0^x f(t)\,dt\right] &= [f(x), x]\left[1, \int_0^x f(t)\,dt\right] \\ &= [f(x), x]\begin{pmatrix} 1 & | & 0 \\ - & | & - \\ 0^T & | & \left[\int_0^x f(t)\,dt\right] \end{pmatrix}. \end{aligned} \qquad (14)$$

Iterating (14) yields

$$\begin{aligned}\left[f(x), \int_0^x f(t)\,dt\right] &= [f(x), x]\begin{pmatrix} 1 & | & 0 \\ - & | & - \\ 0^T & | & \overrightarrow{[f(x), x]} \end{pmatrix} \\ &= \overrightarrow{[f(x), x]}\end{aligned}$$

by (2).

(ii) By (6), the inverse Riordan array

$$\begin{aligned}\left(\overleftarrow{[f(x), x]}\right)^{-1} &= \overrightarrow{([f(x), x]^{-1})} \\ &= \overrightarrow{\left[\frac{1}{f(x)}, x\right]} \\ &= \left[\frac{1}{f(x)}, \int_0^x \frac{dt}{f(t)}\right]\end{aligned}$$

by part (i).

Hence,

$$\overleftarrow{[f(x), x]} = \left[\frac{1}{f(x)}, \int_0^x \frac{dt}{f(t)}\right]^{-1}$$
$$= \left[f\left(g^{<-1>}(x)\right), g^{<-1>}(x)\right], \quad (15)$$

where

$$g(x) = \int_0^x \frac{dt}{f(t)}. \quad (16)$$

Differentiating the identity $g(g^{<-1>}(x)) = x$ with respect to x yields, by the chain rule,

$$\frac{dg^{<-1>}}{dx}(x) = \frac{1}{\frac{dg}{dx}(g^{<-1>}(x))} = f\left(g^{<-1>}(x)\right),$$

since, by (16), $\dfrac{dg(x)}{dx} = \dfrac{1}{f(x)}$.

Thus, by (15),

$$\overleftarrow{[f(x), x]} = \left[\frac{dg^{<-1>}}{dx}(x), g^{<-1>}(x)\right]$$
$$= \left[F(x), \int_0^x F(t)\, dt\right]$$

belongs to the derivative subgroup of the exponential Riordan group, where

$$F(x) = \frac{dg^{<-1>}}{dx}(x) = \frac{d}{dx}\left(\left(\int_0^x \frac{dt}{f(t)}\right)^{<-1>}\right)$$

by (16). ■

Example 5.1. *A094587, the triangle of permutation coefficients, is the exponential Riordan array* $\left[\frac{1}{1-x}, x\right]$.

By Theorem 2 (i), $\overrightarrow{A094587} = \left[\frac{1}{1-x}, log\left(\frac{1}{1-x}\right)\right] = S1$, *the triangle of unsigned Stirling numbers of the first kind A130534.*

$$\begin{pmatrix}1&0&0&0&\dots\\1&1&0&0&\dots\\2&2&1&0&\dots\\6&6&3&1&\dots\\\vdots&\vdots&\vdots&\vdots&\ddots\end{pmatrix}\begin{pmatrix}1&0&0&0&\dots\\0&1&0&0&\dots\\0&1&1&0&\dots\\0&2&2&1&\dots\\\vdots&\vdots&\vdots&\vdots&\ddots\end{pmatrix}\begin{pmatrix}1&0&0&0&\dots\\0&1&0&0&\dots\\0&0&1&0&\dots\\0&0&1&1&\dots\\\vdots&\vdots&\vdots&\vdots&\ddots\end{pmatrix}\dots=\begin{pmatrix}1&0&0&0&\dots\\1&1&0&0&\dots\\2&3&1&0&\dots\\6&11&6&1&\dots\\\vdots&\vdots&\vdots&\vdots&\ddots\end{pmatrix}$$

Example 5.2. *Let $M = [1 + x, x]$. Then, in the notation of Theorem 2 (ii),*

$$\overleftarrow{M} = \left[F(x), \int_0^x F(t)\,dt\right],$$

where

$$F(x) = \frac{d}{dx}\left(\left(\int_0^x \frac{dt}{1+t}\right)^{<-1>}\right) = e^x.$$

Hence $\overleftarrow{M} = [e^x, e^x - 1]= S2$, the triangle of Stirling numbers of the second kind.

$$\dots\begin{pmatrix}1&0&0&0&\dots\\0&1&0&0&\dots\\0&0&1&0&\dots\\0&0&1&1&\dots\\\vdots&\vdots&\vdots&\vdots&\ddots\end{pmatrix}\begin{pmatrix}1&0&0&0&\dots\\0&1&0&0&\dots\\0&1&1&0&\dots\\0&0&2&1&\dots\\\vdots&\vdots&\vdots&\vdots&\ddots\end{pmatrix}\begin{pmatrix}1&0&0&0&\dots\\1&1&0&0&\dots\\0&2&1&0&\dots\\0&0&3&1&\dots\\\vdots&\vdots&\vdots&\vdots&\ddots\end{pmatrix}=\begin{pmatrix}1&0&0&0&\dots\\1&1&0&0&\dots\\1&3&1&0&\dots\\1&7&6&1&\dots\\\vdots&\vdots&\vdots&\vdots&\ddots\end{pmatrix}$$

6. q -analogues of arrays and sequences

A q-analogue of a sequence of numbers is typically a sequence of polynomials in q that reduces to the number sequence when $q = 1$, and satisfies similar algebraic properties, such as recurrences, to the number sequence. One application of the forward and backward arrow operators is to produce q-analogues of sequences and arrays.

Let M be a lower triangular array. Suppose $M(q)$ is a q-analogue of M. Then $\overrightarrow{M(q)}$ is a candidate for a q-analogue of $\overrightarrow{M}$.

$$\begin{array}{ccccc} & M & - \; - \; \rightarrow & \overrightarrow{M} & \\ & | & & | & \\ q-analogue & | & & | & q-analogue \\ & \downarrow & & \downarrow & \\ & M(q) & - \; - \; \rightarrow & \overrightarrow{M(q)} & \end{array}$$

Similarly, $\overleftarrow{M(q)}$ is a candidate for a q-analogue of $\overleftarrow{M}$.

Example 6.1. *Let $U(q)$ denote the Riordan array $\left(\frac{1}{1-x}, qx\right)$ regarded as a q-analogue of the Riordan array $U = \left(\frac{1}{1-x}, x\right)$. Then $\overrightarrow{U(q)}$ is a q-analogue of Pascal's triangle $P = \overrightarrow{U}$. The first few rows of $\overrightarrow{U(q)}$ are shown below.*

$$\begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ 1 & q & 0 & 0 & \dots \\ 1 & q & q^2 & 0 & \dots \\ 1 & q & q^2 & q^3 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & \dots \\ 0 & 1 & q & 0 & \dots \\ 0 & 1 & q & q^2 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & \dots \\ 0 & 0 & 1 & q & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \dots = \begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ 1 & q & 0 & 0 & \dots \\ 1 & q\begin{bmatrix} 2 \\ 1 \end{bmatrix}_q & q^3 & 0 & \dots \\ 1 & q\begin{bmatrix} 3 \\ 1 \end{bmatrix}_q & q^3\begin{bmatrix} 3 \\ 2 \end{bmatrix}_q & q^6 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

where $\begin{bmatrix} n \\ k \end{bmatrix}_q$ is a q-binomial coefficient.

The row generating polynomials of $\overrightarrow{U(q)}$ factorise into linear factors and give a q-analogue of the Binomial Theorem:

$$\prod_{i=1}^{n} (1 + q^i z) = \sum_{k=0}^{n} q^{k(k+1)/2} \begin{bmatrix} n \\ k \end{bmatrix}_q z^k.$$

Example 6.2. *As we saw in Example 2.1, $\overleftarrow{U}$ is the Catalan triangle A033184, whose first column is the sequence of Catalan numbers. Thus the first column of $\overleftarrow{U(q)}$, which begins*

$$[1, 1, 1 + q, 1 + 2q + q^2 + q^3, 1 + 3q + 3q^2 + 3q^3 + 2q^4 + q^5 + q^6, ...],$$

is a candidate for a q-analogue of the Catalan numbers.

These polynomials appear to be the area generating functions $C_n(q)$ of Dyck paths introduced by Carlitz and Riordan. See [2, Proposition 1.6.1, p. 8]. They satisfy the recurrrence

$$C_n(q) = \sum_{k=1}^{n-1} q^{k-1} C_k(q) C_{n-k}(q), \; n \geq 2,$$

with $C(1, q) = 1$.

References

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