function (i.e., $f(m_1 m_2) = f(m_1)f(m_2)$ for coprime m_1, m_2) so a Dirichlet series $\Phi(s)$ is an appropriate generating function. We find $f(p^r) = 2$ for a prime power p^r ($p \equiv 1 \pmod 4$, $r \geq 1$) and obtain

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$$\begin{aligned}\Phi(s) &= \sum_{m=1}^{\infty} \frac{f(m)}{m^s} \\ &= \prod_{p\equiv 1(4)} \left(1 + \frac{2}{p^s} + \frac{2}{p^{2s}} + \cdots\right) = \prod_{p\equiv 1(4)} \frac{1+p^{-s}}{1-p^{-s}} \\ &= 1 + \frac{2}{5^s} + \frac{2}{13^s} + \frac{2}{17^s} + \frac{2}{25^s} + \frac{2}{29^s} + \frac{2}{37^s} \\ &\quad + \frac{2}{41^s} + \frac{2}{53^s} + \frac{2}{61^s} + \frac{4}{65^s} + \frac{2}{73^s} + \cdots .\end{aligned}$$

This generating function is not only a succinct way of representing the statistics of CSL indices, it is also a powerful tool for determining their asymptotic properties [11]. For example, we have used it to show that the number of CSL rotations of $\mathbb{Z}^2$ with index $< X$ is asymptotically $4X/\pi$. The possible CSL indices are precisely the numbers m with all prime factors $\equiv 1 \pmod 4$ and we have $f(m) = 2^a$, where a is the number of distinct prime divisors of m. Each CSL is itself a square lattice, with the index as the area of its fundamental domain.

1.2. Coincidence rotations for sixfold symmetry

Before treating twelvefold symmetry (the main aim of Part 1 of this article) we look at sixfold symmetry. The triangular (or hexagonal) lattice consists of all integral linear combinations of the two vectors e_1 and $\frac{1}{2}(e_1 + \sqrt{3}\, e_2)$. It is (up to a scale factor $\sqrt{2}$) the root lattice A_2 [15]. A rotated copy RA_2 with $R \in \mathrm{SO}(2)$ results in a CSL of finite index if and only if $\cos(\varphi) \in \mathbb{Q}$ and $\sin(\varphi) \in \sqrt{3}\,\mathbb{Q}$. This defines $\mathrm{SOC}(A_2)$ as a subgroup of $\mathrm{SO}(2, \mathbb{Q}(\sqrt{3}))$. To further describe $\mathrm{SOC}(A_2)$ we notice that A_2 can be written as

$$\frac{1}{\sqrt{2}} A_2 = \{m + n\varrho \mid m, n \in \mathbb{Z}\} = \mathbb{Z}[\varrho] \qquad (8)$$

with $\varrho = \frac{1}{2}(1 + i\sqrt{3})$. The lattice $A_2/\sqrt{2}$ is therefore the ring of integers of the imaginary quadratic field $\mathbb{Q}(\sqrt{-3})$, the so-called Eisenstein (or Eisenstein-Jacobi) integers [14]. It has a finite group of units isomorphic to C_6 (namely, ϱ and its powers) and unique prime factorization up to units.

A rotation $R(\varphi) \in \mathrm{SOC}(A_2)$ corresponds to multiplication by a complex number $e^{i\varphi} \in \mathbb{Q}(\sqrt{-3})$. That number can be written as $e^{i\varphi} = \alpha/\beta$ with $\alpha, \beta \in \mathbb{Z}[\varrho]$ coprime and of equal norm. As a consequence of the unique factorization one can again show [11] that every coincidence rotation can be factorized, this time as

$$e^{i\varphi} = \varepsilon \cdot \prod_{p\equiv 1\,(3)} \left(\frac{\omega_p}{\overline{\omega}_p}\right)^{n_p} \qquad (9)$$

where $n_p \in \mathbb{Z}$ (only finitely many of them $\neq 0$), ε is a unit in $\mathbb{Z}[\varrho]$ (a power of ϱ), p runs through the rational primes congruent to 1 (mod 3) and the $\omega_p, \overline{\omega}_p$ are the (complex conjugate) Eisenstein prime factors of p (i.e., $\omega_p \overline{\omega}_p = p$). We thus have

$$\begin{aligned}\mathrm{SOC}(A_2) = \{R \in \mathrm{SO}(2) \mid \cos(\varphi) \in \mathbb{Q},& \\ \sin(\varphi) \in \sqrt{3}\,\mathbb{Q}\}& \qquad (10) \\ \simeq C_6 \otimes \mathbb{Z}^{\aleph_0}&\end{aligned}$$

with generators ϱ for C_6 and $\omega_p/\overline{\omega}_p$ with $p \equiv 1\ (3)$ for the infinite cyclic groups.

As in the previous example, the coincidence index is 1 for the units and p for the other generators, so for a rotation $R(\varphi)$ factorized as in (9), we have

$$\Sigma(R) = \prod_{p\equiv 1\,(3)} p^{|n_p|} . \qquad (11)$$

The first three generators with $\Sigma > 1$, normalized to have denominator Σ (a prime $\equiv 1$ (mod 3)) and argument in $(0, \pi/6)$, are

$$\frac{5+3\varrho}{7}, \ \frac{8+7\varrho}{13}, \ \frac{16+5\varrho}{19}.$$

Finally, if $6f(m)$ denotes the number of CSL rotations of index m, $f(m)$ is multiplicative and one finds the Dirichlet series generating function [11]

$$\begin{aligned}\Phi(s) = \sum_{m=1}^{\infty} \frac{f(m)}{m^s} &= \prod_{p\equiv 1\,(3)} \frac{1+p^{-s}}{1-p^{-s}} \\ &= 1 + \frac{2}{7^s} + \frac{2}{13^s} + \frac{2}{19^s} + \frac{2}{31^s} + \frac{2}{37^s} + \frac{2}{43^s} \\ &\quad + \frac{2}{49^s} + \cdots + \frac{2}{79^s} + \frac{4}{91^s} + \frac{2}{97^s} + \cdots\end{aligned}$$

The possible coincidence indices are precisely the numbers m with all prime factors $\equiv 1$ (mod 3) and

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