Proof of a conjecture stated in A026641

Sela Fried

Sequence A026641 is defined as the number of nodes of even outdegree (including leaves) in all ordered trees with n edges. The purpose of this note is to prove the following result, conjectured by Peter Bala.

Theorem 1. *For an integer $n \geq 0$, let $a(n) =$ A026641(n). Suppose that $a(n)$ is odd. Then $a(n) \equiv 1 \pmod 4$.*

Proof. According to Bala,

$$a(n) = \sum_{k=0}^{n} \binom{2n+1}{n+k+1}(-2)^k. \tag{1}$$

Furthermore, Bala stated that $a(n)$ is odd if and only if $n = 2^k - 1$ for some nonnegative integer k. Thus, we consider n of this form. Notice that, if $k = 0$, then $n = 0$ and the assertion holds trivially. Thus, we assume that $k \geq 1$. Reducing (1) modulo 4, we obtain

$$a(n) \equiv \binom{2n+1}{n+1} - 2\binom{2n+1}{n+2} \pmod 4. \tag{2}$$

Thus, it suffices to prove that

$$\binom{2n+1}{n+1} \equiv 3 \pmod 4, \tag{3}$$

$$\binom{2n+1}{n+2} \equiv 1 \pmod 4. \tag{4}$$

We have,

$$\binom{2n+1}{n+1} = \binom{2^{k+1}-1}{2^k} = \prod_{j=1}^{2^k-1} \frac{2^k+j}{j}.$$

Let $P = \prod_{j=1}^{2^k-1} \frac{2^k+j}{j}$ and write each j as $j = 2^{\ell_j} u_j$ with $0 \leq \ell_j \leq k-1$ and u_j odd. Thus,

$$P = \frac{\prod_{j=1}^{2^k-1}(2^{k-\ell_j} + u_j)}{\prod_{j=1}^{2^k-1} u_j}.$$

The denominator $\prod_{j=1}^{2^k-1} u_j$ is odd and therefore invertible modulo 4. Hence,

$$P \equiv \left(\prod_{j=1}^{2^k-1} (2^{k-\ell_j} + u_j) \right) \left(\prod_{j=1}^{2^k-1} u_j \right)^{-1} \pmod{4}.$$

Now, if $k - \ell_j \geq 2$, then $2^{k-\ell_j} + u_j \equiv u_j \pmod{4}$. On the other hand, $k - \ell_j = 1$ if and only if $j = 2^{k-1}$ (i.e., $u_j = 1$) and therefore, for this j, $2^{k-\ell_j} + u_j \equiv 3 \pmod{4}$. This proves (3).

We now prove (4). If $k = 1$, then the identity holds trivially. Assume that $k \geq 2$. We have,

$$(n+2)\binom{2n+1}{n+2} = n\binom{2n+1}{n+1}. \tag{5}$$

Furthermore, $n = 2^k - 1 \equiv 3 \pmod{4}$ and $n + 2 = 2^k + 1 \equiv 1 \pmod{4}$. Reducing (5) modulo 4 and using (3) we obtain

$$\binom{2n+1}{n+2} \equiv 3\binom{2n+1}{n+1} \equiv 3 \cdot 3 \equiv 1 \pmod{4},$$

as asserted. □

References

[1] N. J. A. Sloane, The On-Line Encyclopedia of Integer Sequences, OEIS Foundation Inc., `https://oeis.org`.