On the morphism $1 \to 121$, $2 \to 12221$

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Dedicated to Neil Sloane, who, in particular, is the creator of the Encyclopedia of Integer Sequences

Abstract

We describe several occurrences of the morphism $1 \to 121$, $2 \to 12221$ and the closely related morphism $2 \to 211$, $1 \to 2$ (as well as simple variants) in the literature. Furthermore we prove that a sequence in the OEIS, proposed by Kimberling, is the same as a sequence independently studied by Akiyama, Brunotte, Pethő, and Steiner related to a conjecture on the periodicity of certain piecewise affine planar maps. Finally we prove conjectures of Kimberling and conjectures of Baysal in the OEIS.

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1 Introduction

A fantastic tool for the study of sequences of integers is the Online Encyclopedia of Integer Sequences (OEIS) [26]. Every time one encounters a sequence of integers, one can (should) look for it in the OEIS: very often, unexpected relations between sequences can be found. If not, it may happen that one finds independently surprising links that will feed the Encyclopedia. Another aspect of the OEIS is that it suggests conjectures it is always fun to work on. In this paper we focus on the sequence A026465 = $1, 2, 1, 1, 2, 2, 2, 1, 1, 2, 1, 1, 2, 1, 1, 2, \ldots$ in the OEIS and closely related sequences.

We begin with a very brief discussion about morphisms of monoids generated by finite sets (for more details, the reader can look at, e.g., [8, 22, 23]). Let $\mathcal{A}$ be a finite set. We let $\mathcal{A}^*$ denote the free monoid generated by $\mathcal{A}$, equipped with the concatenation operation. The elements of $\mathcal{A}^*$ are called words over the alphabet $\mathcal{A}$. If $\mathcal{A}$ and $\mathcal{B}$ are two finite sets, a morphism (i.e., a monoid morphism) from $\mathcal{A}^*$ to $\mathcal{B}^*$ is a map from $\mathcal{A}^*$ to $\mathcal{B}^*$ that preserves the concatenation. Such a morphism is defined by its values on $\mathcal{A}$. A morphism from $\mathcal{A}^*$ to itself is called a morphism of $\mathcal{A}^*$. The union of $\mathcal{A}^*$ and $\mathcal{A}^{\mathbb{N}}$ (the sequences—also called infinite words—on $\mathcal{A}$ indexed by $\mathbb{N} = 0, 1, 2, \ldots$ —or sometimes by $1, 2, \ldots$) is equipped with the topology of simple convergence (two words or sequences are "close" if they coincide on a "long" prefix).

In what follows we will be chiefly interested in the morphism ν of $\{1, 2\}^*$ defined by

$$\nu(1) := 121, \quad \nu(2) := 12221.$$

Iterating this morphism, starting from 1, we obtain successively

$$\begin{aligned}
\nu^0(1) &= 1 \\
\nu^1(1) &= 121 \\
\nu^2(1) &= 12112221121 \\
&\vdots
\end{aligned}$$

When the number of iteration tends to infinity, the sequence of words $1, 121, 12112221121, \ldots$ tends to an infinite sequence $N = 1211222112112112221 \cdots$. This is sequence A026465 in the OEIS.

In the sequel we will give several (sometimes unexpected) properties of the sequence N and of a cousin sequence P defined in Section 2 below. Some of these properties are chosen from those given in the OEIS, some others are not (yet) in the OEIS. Furthermore we will show that sequence A260456 in the OEIS [26] is related to sequence N. Finally we will prove conjectures of Kimberling and conjectures of Baysal given in the OEIS.

2 A cousin of the sequence N

Instead of the morphism ν above, consider the morphism φ defined on $\{1, 2\}^*$ by

$$\varphi(2) := 211, \quad \varphi(1) := 2.$$

Iterating φ starting from 2, we obtain successively

$$\begin{aligned}
\varphi^0(2) &= 2 \\
\varphi^1(2) &= 211 \\
\varphi^2(2) &= 21122 \\
\varphi^3(2) &= 21122211211 \\
&\vdots
\end{aligned}$$

When the number of iteration tends to infinity, the sequence of words $211, 21122, 21122211211, \ldots$ tends to an infinite sequence $P := 211222112112112221 \cdots$

It happens that N and P are very closely related as indicated in the classical proposition below. (For the sake of completeness we give a proof inspired by the proof proposed in [6].)

Proposition 1 *The sequence N is obtained by padding a 1 in front of the sequence P. In other words we have $N = 1P$.*

Proof. First, we note that, for any word x, we have $21\nu(x) = \varphi^2(x)21$. It suffices to prove this equality for $x = 1$, $x = 2$, which is immediate, and to extend it to any word by morphicity. Now, applying this equality to the prefix of N of length k, say $N_{(k)}$, we obtain $21\nu(N_{(k)}) = \varphi^2(N_{(k)})21$. Letting k go to infinity, this gives $21\nu(N) = \varphi^2(N)$. Since $\nu(N) = N$, this can be written $21N = \varphi^2(N)$. Defining B by $N = 1B$, we have

$$211B = \varphi^2(1B) = \varphi^2(1)\varphi^2(B) = 211\varphi^2(B) \text{ hence } B = \varphi^2(B).$$

This means that B is a fixed point of φ^2. But P, being a fixed point of φ is also a fixed point of φ^2. Since φ^2 clearly admits only one fixed point, one has $B = P$. $\square$

Remark 2 Morphisms like ν and φ, namely for which there exists a fixed word, say z (here $z = 21$), such that for all words x one has $z\nu(x) = \varphi(x)z$, are called *conjugate*. If x_n is a sequence of words that converges to an infinite word X, it is clear that the equalities $z\nu(x_n) = \varphi(x_n)z$ for all n imply $z\nu(X) = \varphi(X)$.

Remark 3 It is clear that the frequencies of 1 (resp. of 2) are the same in N and in P. Furthermore the *incidence matrix* (see, e.g., [8, Section 8.2, p. 248]) of ν is $\begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix}$. It is primitive and positive, and its Perron-Frobenius eigenvalue is equal to 4; the associated normalized positive vector is the vector $\begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}$. Hence the frequencies of 1 and 2 in N (and in P) are equal to 1/2 (see, e.g., [8, Th. 8.4.6 and Th. 8.4.7, pp. 271–272]).

3 Relation with the (Prouhet-)Thue-Morse sequence

The Prouhet-Thue-Morse is a famous binary sequence that occurs in many contexts (see, e.g., [7]; also see A010060 in the OEIS). One of the definitions of this sequence is that it is the fixed point beginning with 0 of the morphism m defined on $\{0, 1\}^*$ by $m(0) := 01$, $m(1) := 10$, so that this sequence is equal to

$$0\ 1\ 1\ 0\ 1\ 0\ 0\ 1\ 1\ 0\ 0\ 1\ 0\ 1\ 1\ \ldots$$

Writing this sequence as 0 11 0 1 00 11 00 1 0 1 1 ... and defining the *runs* of this sequence as the consecutive lengths of maximal blocks containing only 0 or only 1, we see that the sequence of runs of the Prouhet-Thue-Morse sequence begins

$$1\ 2\ 1\ 1\ 2\ 2\ 2\ 1\ 1\ \ldots$$

This sequence of 1 and 2 is described in [11] (see page 86 and page 88), where it is used to compute the block-complexity of the Thue-Morse sequence (i.e., the number of blocks of each length in the Thue-Morse sequence). The fact that the sequence of runlengths of the Thue-Morse sequence is precisely N appears, e.g., in [5, pp. 306–307], [4, p. 458], [25, p. 354], [3, p. 2122], and in [26, sequence A026465; 2019-Dekking's comment]:

Proposition 4 *The sequence N, fixed point of the morphism ν, is the sequence of runlengths of the Prouhet-Thue-Morse sequence.*

This well-known statement can be proved, e.g., by using the locally catenative properties of morphic sequences (for this aspect of morphic sequences, see [24]).

4 "Vile" integers and the sequence $P = 2, 1, 1, 2, 2, 2, 1, 1, \ \ldots$

Recall that Fraenkel [16, p. 43] called "vile" the integers whose binary representation ends in an even number of zeros. The increasing sequence of vile numbers is the sequence A003159 in [26]

$$1, 3, 4, 5, 7, 9, 11, 12, 13, 15, 16, 17, 19, 20, 21, \ldots$$

Among the marvellous properties of this sequence, we only note here that twice this sequence, namely the "dopey" numbers [16, p. 43]

$$2, 6, 8, 10, 14, 18, 22, 24, 26, 30, 32, 34, 38, 40, 42, \ldots$$

and the sequence itself from a partition of the positive integers. Furthermore the sequence can be linked to the names of the first ten integers in Hungarian and in Japanese (see [28])! Note that this sequence is also studied in [4].

Perhaps unexpectedly sequence A003159 is related to P. Namely, as indicated by Barry in the comments on this sequence in [26]:

Theorem 5 (Barry) *The first difference of sequence* A003159 *is sequence P. (Equivalently* A003159 *is the summatory function of* A026465*.)*

5 The morphism φ, its fixed point P, and Sturmian sequences

Recall that for a sequence with values in the alphabet $\mathcal{A}$, its *recurrence function* $n \rightarrowtail R(n)$ is the size of the smallest window containing an occurrence of each block of this sequence, whatever its position is (the value $R(n)$ value can be infinite). The *recurrence quotient* of the sequence is defined by $\rho := \limsup_{n\to\infty} \frac{R(n)}{n}$ (it can also be infinite). Finally we recall that a Sturmian sequence is a sequence defined on the alphabet $\{0,1\}$, that has exactly $n+1$ distinct blocks of length n. (For more about these definitions, in particular for a geometric definition of Sturmian sequences, one can consult, e.g., [8, 18].) Cassaigne [13] proved in 1999 the following theorem, involving the sequence P (the fixed point of φ seen previously):

Theorem 6 (Cassaigne) *Let* $[\mathbf{u}]$ *be the real number whose continued fraction expansion is given by*

$$[\mathbf{u}] := [2,1,1,2,2,2,1,1,2,\dots]$$

where $211222112\dots$ *is sequence* P*. Then, the smallest accumulation point of the set of recurrence quotients of all Sturmian sequences is equal to* $2+[\mathbf{u}]$*.*

6 A conjecture of Mahler and a result of Dubickas

Mahler [20] defined Z-numbers as positive real numbers such that their fractional parts $\{\alpha(\frac{3}{2})^n\}$ belong to $[0,1/2)$, for all $n \geq 0$. Quoting [20]:

> Several years ago, a Japanese colleague proposed to me the problem whether such Z-numbers do in fact exist. I have not succeeded in solving this problem, but shall give here a number of incomplete results. In particular, it will be proved that *the set of all Z-numbers is at most countable.*

The question whether Z is empty or not, known as Mahler conjecture, is still open. Among the numerous partial related results that have been obtained, one is curiously linked to the sequence $N = 1,2,1,1,2,2,2,1,1,2,1,1\dots$ Namely, let $W(x) := 1+2x+x^2+x^3+2x^4+\cdots$ be the generating function of sequence N. Also let $F(x) := W(-x)$. Dubickas proved [15] the following result for *negative* rationals (also see [14] for related results).

Theorem 7 (Dubickas) *Let* $b = -p/q$*, where* $p > q > 1$ *are two coprime positive integers. Then, for any real number* ξ*, the sequence of fractional parts* $\{\xi b^n\}$*,* $n = 0,1,2,\dots$*, has a limit point* $\leq 1-(1-F(q/p))/q$*; also, if* $\xi \neq 0$ *then it has a limit point* $\geq 1-F(q/p))/q$*. In particular, for any real number* $\xi \neq 0$*, the sequence of fractional parts* $\{\xi(-3/2)^n\}$*,* $n = 0,1,2,\dots$*, has a limit point smaller than* 0.533547 *and a limit point greater than* 0.466452*.*

7 Sum-free sets and a family of morphism generalizing φ

Recall that a set S is called sum-free if $S \cap (S+S) = \emptyset$, where $S+S := \{x+y \mid x,y \in S\}$. Without entering technical details, we indicate that the authors of [9] study sum-free sets generated by "period-k-folding sequences". The first differences of these sequences are morphic. For $k=1$, the morphism is precisely φ.

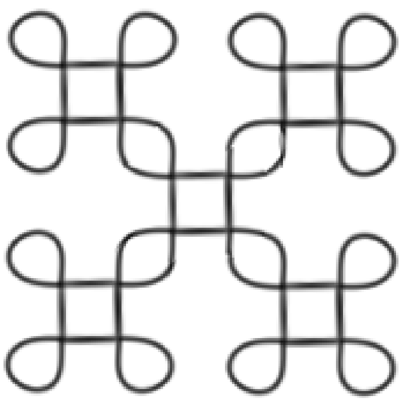
8 Dimensions of the lower central series factors of a certain just infinite Lie algebra

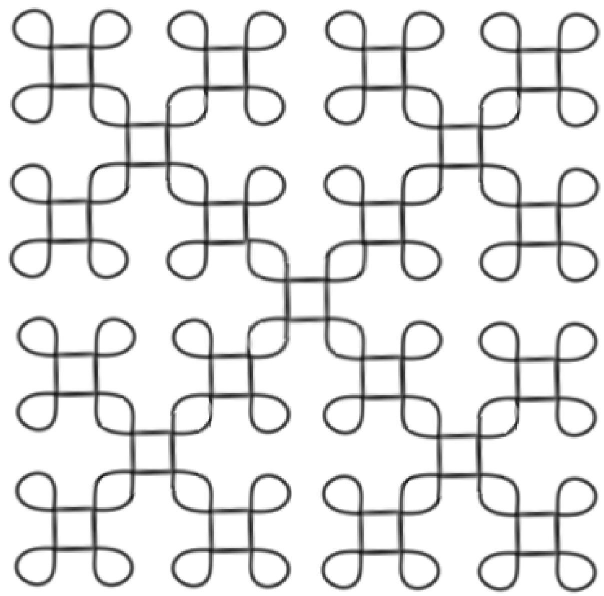
The following result was conjectured by Dekking and proved in [6]. It involves the morphism ν defined above (recall that $\nu(1) := 121$, $\nu(2) := 12221$).

Theorem 8 (Allouche-Petrogradsky) *Let* $\mathbf{R}$ *be the* 2*-generated just infinite nil graded fractal Lie superalgebra defined in [21] in the case of characteristic* 2*. By discarding a few elements, one obtains a Lie algebra also denoted by* $\mathbf{R}$*. Then, the sequence* $(a_n)_{n\geq 1}$*, defined by* $A = (a_n)_{n\geq 1} := \dim \mathbf{R}^n/\mathbf{R}^{n+1} = \dim \mathbf{R}_n$ *satisfies the property* $A = 1221\nu(A)$*.*

9 The morphism φ and a classical Kolam

The art of *kolam* (or *kalam*, or *rangoli*) is a traditional decorative art of pictures drawn on the ground with rice flour. Present in India, kolam, or variations thereof, can be found in other countries (e.g., in the Vanuatu Islands). To give a pictorial idea of a classical kolam, we only show the drawings below: this classical kolam can be actually drawn by using the morphism φ: see [3]. The reader can consult references given in [3] for more about kolam (and ethnomathematics).





10 A simple variant of N and the period-doubling sequence

Sequence A080426 in [26] is a variant of sequence N considered above. Namely, it is the fixed point of $1 \to 131$; $3 \to 13331$. Thus it is identical to the sequence N considered above, after replacing in N all 3's with 2's. This sequence A080426 has several properties from which we select the following ones.

First we give a proof of a remark of Hofstadter in [26, A080426] applied to N.

Proposition 9 *Let α be the morphism defined by $\alpha(1) := 2$, $\alpha(2) := 112$. Define a sequence of words S_n by $S_0 := 1$, and, for all $n \geq 0$, $S_{n+1} := \alpha(S_n)$. Then $N = S_0 S_1 S_2 \cdots S_n \cdots$.*

Proof. To begin with, note that the sequence $T := S_0 S_1 \cdots S_n \cdots$ satisfies $T = 1\alpha(T)$. Recall that $N = 1P$, and that N (resp. P) is the fixed point of ν (resp. φ) defined in the introduction. First, we prove that for all words $x \in \{1,2\}^*$ one has $\alpha\nu(x) = \varphi^2\alpha(x)$ (it suffices to prove this for $x = 1$ and $x = 2$ and to use morphicity). As previously one can take for x an infinite word, in particular for N we obtain $\alpha\nu(N) = \varphi^2\alpha(N)$. But $\nu(N) = N$, thus $\alpha(N) = \varphi^2(\alpha(N))$. This implies that $\alpha(N)$ is a fixed point of φ^2, which admits a unique fixed point, the fixed point of φ, i.e., P. Thus we have $\alpha(N) = P$. Remembering that $N = 1P$ this yields $N = 1P = 1\alpha(N)$. Thus N and T satisfy the same equality $X = 1\alpha(X)$. But only one sequence satisfies this equality: namely for such a sequence, $0X$ is the unique fixed point of the morphism $\widetilde{\alpha}$ defined on $\{0,1,2\}$ by $\widetilde{\alpha}(0) := 01$, $\widetilde{\alpha}(1) := \alpha(1)$, $\widetilde{\alpha}(2) := \alpha(2)$. $\square$

Now, we give relations between A080426 and A035263. Recall that sequence A035263 in [26], the period-doubling sequence (also called the Feigenbaum sequence, as it occurs in the study of Feigenbaum cascades in the iteration of unimodal functions of the interval) is defined as the fixed point of the morphism $1 \to 10$, $0 \to 11$.

Proposition 10 (Deléham, OEIS) *Sequence* A080426 *is the length of the n-th run of* 1*'s in the period-doubling sequence.*

Remark 11 This link between A080426 and the period-doubling sequence is essentially the same as the statement given in [10, p. 4] (with different notation): *Let τ be the morphism* $1 \to 10$, $3 \to 1110$. *Then the image by τ of* A080426 *is the period-doubling sequence.*

Remark 12 Since the period-doubling sequence has no occurrence of 00 in it, the length of the n-th run of 1's in this sequence is also the *gap* between two occurrences of 0 (considering that the first gap occurs just before the occurrence of the first 0). It would be interesting to look at the sequence of gaps between the successive occurrences of any given block: this has been done for the Thue-Morse sequence in [27].

Deléham states another result in the comments about sequence A003156, which shows that A003156 is the summatory function of A080426.

Proposition 13 (Deléham, OEIS) *Sequence* A003156 *is equal to the number of ones before the n-th zero in the period-doubling sequence. (Thus, it is equal to the summatory function of* A080426*.)*

In view of Propositions 10 and 13, one can ask whether there is also a similarity between the original construction of A003156 and the sequence A080426. First, let us give the original construction of A003156.

Define three sequences $(A_n)_{n\geq 1}$, $(B_n)_{n\geq 1}$, $(C_n)_{n\geq 1}$ by: $A_1 := 1$, $B_1 := 3$, $C_1 := 2$, and, for $n \geq 2$,

$$A_n := \operatorname{mex}\{A_i, B_i, C_i \mid i < n\}, \quad B_n := A_n + 2n, \quad C_n := B_n - 1$$

(for a set of integers S, $\operatorname{mex}(S)$ is the least integer not in S). A003156 is by definition the sequence $(A_n)_{n\geq 1}$.

These three sequences are studied in [12], where it is proved in particular that $(A_n)_{n\geq 1}$, $(B_n)_{n\geq 1}$, $(C_n)_{n\geq 1}$ form a partition of the positive integers. Interestingly enough an expression for the summatory function of the vile numbers (defined in Section 4 above) similar to the definition above can be found in [17], namely:
Let $A'_1 = 1$, $B'_1 = 2$, $C'_1 = 3$. For $n \geq 2$, let

$$A'_n := \operatorname{mex}\{A'_i, B'_i, C'_i \mid i < n\}, \quad B'_n := \operatorname{mex}(\{A'_i, B'_i, C'_i \mid i < n\} \cup \{A_n\}, \quad C'_n := A'_n + B'_n.$$

Let $(D_k)_k$ be the increasing sequence of all integers n for which $B'_n = A'_n$.

It is not hard to see that $\frac{1}{4}D_k$ is precisely the sequence of vile numbers A003159 (see Section 4 above; also see [4]).

Remark 14 The reader will have found the relation $2 \cdot \text{A003159}(n) - n = \text{A003156}(n)$, for $n \geq 1$, (given by Deléham in the OEIS for A003156) which could be used to explain the similarity of the two definitions involving "mex". But, more generally, it is not clear what a general family of sequences defined in an analogous way could be.

11 Another simple variant of sequence N

Another simple variant of N that can be found in the literature is obtained by replacing in N all 1's by 0's and all 2's by 1's.

11.1 A conjecture of Akiyama, Brunotte, Pethő and Thuswaldner, and Vivaldi

A conjecture in [2, Conjecture 6.1] and [29] stipulates that: *For every real λ with $|\lambda| < 2$, all integer sequences $(a_k)_{\in\mathbb{Z}}$ satisfying $0 \le a_{k-1} + \lambda a_k + a_{k+1} < 1$ are periodic.* Particular cases of this conjecture have been addressed in the literature, e.g., the computer assisted proof for the case where λ is the golden ratio [19]. In 2008, Akiyama, Brunotte, Pethő and Steiner [1] addressed the cases $\lambda \in \{\frac{\pm 1 \pm \sqrt{5}}{2}, \pm\sqrt{2}, \pm\sqrt{3}\}$ in an ingenious geometrico-combinatorial proof: part of their proof use an avatar of the sequence N above, namely the fixed point of the morphism $0 \to 010$, $1 \to 01110$, i.e., the sequence

$$\widetilde{N} := 0\ 1\ 0\ 0\ 1\ 1\ 1\ 0\ 0\ 1\ 0\ 0\ 1\ 0\ 0\ 1\ 1\ 1\ 0\ 0\ \ldots$$

(Note that this sequence can clearly be obtained by subtracting 1 from every term of N. Also see next subsection.) The authors of [1] also consider the image of $\widetilde{N}$ by the morphism $0 \to 10$, $1 \to 110$ (see [1, p. 241]): let us call this sequence $N^{\#}$. It happens that sequence $N^{\#}$ is equal to a sequence in the OEIS independently proposed by Kimberling in 2015, namely A260456.

Theorem 15 *Sequence* A260456[1] *in the OEIS is equal to* $N^{\#}$

Proof. First we we recall that N is the fixed point of the morphism ν, where $\nu(1) = 121$, $\nu(2) = 12221$. Also we note that $N^{\#}$ can clearly be defined as the image of sequence N by the morphism θ defined by $\theta(1) = 10$, $\theta(2) = 110$.

Sequence A260456 in the OEIS is defined as the limit of the sequence of words $(w_n)_{n\ge1}$, where $w_1 := 1$, and, for all $n \ge 0$, $w_{2n} := w_{2n-1}0w_{2n-1}$ and $w_{2n+1} := w_{2n}w_{2n}$. Letting $z_n := w_{2n+1}$, we clearly have $z_0 = 1$, and, for all $n \ge 0$, $z_{n+1} = z_n0z_nz_n0z_n$. Furthermore the sequence of words $(z_n)_{n\ge0}$ tends to A260456.

Now, we claim that, for all $n \ge 0$, we have

$$\theta(\nu^n(1)) = z_n0 \quad\text{and}\quad \theta(\nu^n(2)) = z_nz_n0.$$

This is proved by induction on n. The property is clearly true for $n = 0$. If it is true for n, then

$$\theta(\nu^{n+1}(1)) = \theta(\nu^n(121)) = \theta(\nu^n(1))\theta(\nu^n(2))\theta(\nu^n(1)) = z_n0z_nz_n0z_n0 = z_{n+1}0$$

and

$$\begin{aligned}\theta(\nu^{n+1}(2)) &= \theta(\nu^n(12221)) = \theta(\nu^n(1))\theta(\nu^n(2))\theta(\nu^n(2))\theta(\nu^n(2))\theta(\nu^n(1)) \\ &= z_n0z_nz_n0z_nz_n0z_nz_n0z_n0 = z_{n+1}z_{n+1}0. \quad \square\end{aligned}$$

11.2 Proofs of conjectures in the OEIS for sequences A284388 and A284391

In what follows we will address conjectures given in the OEIS for sequences A284388 and A284391 These sequences take their values in $\{0, 1\}$. We could have changed the statements of conjectures and results in this section by replacing in both sequences all 0' by 1's and all 1's by 2's to stick to our previous friend N, but we have chosen to work with the original statements. Hence we introduce the morphism $\widetilde{\nu}$ defined on $\{0, 1\}$ by $\widetilde{\nu}(0) := 010$, $\widetilde{\nu}(1) := 01110$, and we call $\widetilde{N}$ its fixed point.

The sequences A284388 and A284391 in [26] are defined as the "limiting words" of the morphism $0 \to 1$, $1 \to 001$. More precisely, A284388 is defined as the "0-limiting word" of this morphism (i.e., as the fixed point beginning with 0 of the square of this morphism); and A284391 is defined as the "1-limiting word" of this morphism (i.e., as the fixed point beginning with 1 of the square of this morphism). Squaring the morphism $0 \to 1$, $1 \to 001$, we obtain the morphism h defined by $h(0) := 001$, $h(1) := 11001$, so that the sequences A284388 and A284391 are respectively the fixed point beginning with 0 and the fixed point beginning with 1 of h. Actually both sequences are closely related to the sequence N, as will be explained now.

[1] The fact that the name "A260456" is an anagram of "A026465", though unexpected, is a pure coincidence.

Theorem 16 *Let $\widetilde{\nu}$ be the morphism defined on $\{0,1\}$ by $\widetilde{\nu}(0) := 010$, $\widetilde{\nu}(1) := 01110$. Let $\widetilde{N}$ be its fixed point. Then, we have the following properties.*

- ∗ *Deleting the first term (which is equal to 0) of* A284388 *yields the sequence $\widetilde{N}$.*
- ∗ *Replacing the first term (which is equal to 1) of* A284391 *with a* 0 *yields the sequence $\widetilde{N}$.*

Proof. The morphism $\widetilde{\nu}$ is clearly, up to notation, the same as the morphism ν above (replace in ν the letter 1 with 0 and the letter 2 with 1). Then we note that $\widetilde{\nu}$ and h are conjugate. Namely,

$$01h(0) = 01001 = \widetilde{\nu}(0)01, \quad 01h(1) = 0111001 = \widetilde{\nu}(1)01.$$

Thus, for all words x one has $01h(x) = \widetilde{\nu}(x)01$. Hence, for any infinite sequence X, one has $01h(X) = \widetilde{\nu}(X)$. So that, if X is one of the fixed points of h, i.e., $h(X) = X$, we have $01X = \widetilde{\nu}(X)$. We distinguish two cases:

$$\begin{cases} \text{if } X = 0Y \quad \text{then } \ 010Y = \widetilde{\nu}(0Y) = 010\widetilde{\nu}(Y), \text{ thus } Y = \widetilde{\nu}(Y); \\ \text{if } X = 1Z \quad \text{then } \ 011Z = \widetilde{\nu}(1Z) = 01110\widetilde{\nu}(Z), \text{ thus } Z = 10\widetilde{\nu}(Z); \text{ hence } 0Z = 010\widetilde{\nu}(Z) = \widetilde{\nu}(0Z). \end{cases}$$

Noting that $\widetilde{N}$ is the only fixed point of $\widetilde{\nu}$, this gives:

$$\begin{cases} \text{if } X = 0Y \text{ (i.e., } X = A284388\text{)}, \quad \text{then } \ \ Y = \widetilde{N}; \\ \text{if } X = 1Z \text{ (i.e., } X = A284391\text{)}, \quad \text{then } \ 0Z = \widetilde{N}. \end{cases} \qquad (1) \qquad \square$$

Now we prove a conjecture of Kimberling (see [26, A284388] and [26, A284391]).

Theorem 17 *The frequency of* 0 *and the frequency of* 1 *in the sequences* A284388 *and* A284391 *are all equal to* 1/2*.*

Proof. The result can be deduced from Eq. (1). Namely, these equalities imply that the frequencies of 0 and 1 in $X = 0Y$ (resp. $X = 1Z$) are equal to those in $\widetilde{N}$. But these frequencies are of course equal to the frequencies of 1 and 2 in N, hence to 1/2 (see Remark 3). Actually one can also give a direct proof, that is exactly the same as the proof given in Remark 3, (taking the morphism h and its incidence matrix). □

It happens that what precedes can lead to proving other conjectures (with a slight correction) proposed by Baysal in [26, A284388] and [26, A284391].

Theorem 18

(i) *Excluding the first two terms of sequence* A284388*, if the runs of* 1*'s of length one are replaced with (a single)* 0*, and the runs of* 1*'s of length three are replaced with (a single)* 1*, we get the same sequence preceded by a* 0*.*

(ii) *The index distance between two consecutive* 1*'s is either one or three. Excluding the first two terms of sequence* A284388*, if distances of one are replaced by* 0*, and distances of three are replaced by* 1*, we get the same sequence.*

(iii) *Excluding the first term of* A284391*, if the runs of* 1*'s of length one are replaced with (a single)* 0*, and the runs of* 1*'s of length three are replaced with (a single)* 1*, we get the same sequence.*

(iv) *The index distance between two consecutive* 1 *is either one or three in* A284391*. Excluding the first term of A284391, if distances of one are replaced with* 0*, and distances of three are replaced with* 1*, we get the same sequence.*

Proof. As we have in Equation (1) above, if A284388 = $0Y$ then $Y = \widetilde{N}$; and if A284391 = $1Z$, then $0Z = \widetilde{N}$. Hence excluding the first two terms of A284388 or excluding the first term of A284391 gives the same sequence, namely the sequence $\widetilde{P}$. Thus, it suffices to prove the first two assertions.

(i) In the proof of Theorem 16 we have seen that if X is sequence A284388, then $X = 0\widetilde{N}$. Thus, excluding the first two terms of sequence A284388 yields the sequence $\widetilde{P}$ obtained from sequence P (the cousin of N in Section 2), by replacing (in P) 1 with 0, and 2 with 1. Thus $\widetilde{P}$ is the fixed point of the morphism $\widetilde{\varphi}$ defined by $\widetilde{\varphi}(1) := 100$, $\widetilde{\varphi}(0) := 1$. Note that $\widetilde{N} = 0\widetilde{P}$. Letting β denote the morphism defined by $\beta(0) := 100$, $\beta(1) := 11100$, assertion (i) means that $\beta(\widetilde{N}) = \widetilde{P}$, i.e., $\beta(0\widetilde{P}) = \widetilde{P}$, which can also be written $100\beta(\widetilde{P}) = \widetilde{P}$. Now, in order to prove this last equality, it suffices to prove that, for all words x, one has $100\beta(x) = \varphi^2(x)100$, since P is a fixed point of φ, hence of φ^2. But this equality is clear for $x = 0$ or $x = 1$, hence for all x by morphicity.

(ii) Recall that excluding the first two terms of sequence A284388 yields $\widetilde{P}$ as indicated above. This sequence, being a fixed point of the morphism $\widetilde{\varphi}$ is also a fixed point of $\widetilde{\varphi}^2$. Since $\widetilde{\varphi}^2(1) = 10011$ and $\widetilde{\varphi}^2(0) = 100$, we see that $\widetilde{P}$ is composed of blocks 10011 and 100, and the letter following each such block must be a 1. Thus, coding the index distances between consecutive 1's by 0 for distances of one, and by 1 for distances of three, is exactly replacing in $\widetilde{P}$ each block 10011 with 100, and each block 100 (followed by another 100) with 1, thus yielding a sequence $\varepsilon_1\varepsilon_2\ldots$ Hence, writing $\widetilde{P} = w_1w_2\cdots$, where w_i is either 10011, or 100 (followed by another 100), we have that $w_i = \widetilde{\varphi}^2(\varepsilon_i)$, where $\varepsilon_i = 0$ if $w_i = 10011$, and $\varepsilon_i = 1$ if $w_i = 100$ (followed by another 100). So that $\widetilde{P} = \widetilde{\varphi}^2(\varepsilon_1\varepsilon_2\ldots)$. But we also have that $\widetilde{P} = \varphi^2(\widetilde{P})$. Since there clearly is only one decomposition of $\widetilde{P}$ into blocks 10011, and 100 (followed by another 100), this implies that $\widetilde{P} = \varepsilon_1\varepsilon_2\ldots$, which is the assertion we wanted to prove. □

Remark 19 As M. Dekking told me after reading a previous version of this paper, one can make the proof of Theorem 18 simpler by using *return words*, or even, in this particular case, by seeing directly that A284388 is composed of blocks 001 and 1 and that the morphism ruling these blocks is quite easy to obtain.

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