A PROPERTY OF THE DIVISORS OF 99

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The *reversal* of a number is the number with its decimal digits reversed. For example, the reversal of 123 is 321. Let us say that a natural number n is *magic* if it divides the reversal of every multiple of n. In symbols,

$$n \text{ is magic} \iff \forall k \in \mathbb{N}, \quad n \mid \operatorname{rev}(kn).$$

We will demonstrate that the only magic numbers are the divisors of 99, i.e. 1, 3, 9, 11, 33, and 99. This answers a question posed by James Tanton.

This property of the divisors of 99 is mentioned in the Online Encyclopedia of Integer Sequences, but I have been unable to locate a proof in the literature.

1. Divisors of 99 are magic

Recall that a number is divisible by 3 if and only if the sum of its digits is divisible by 3. Since the sum of the digits is not altered by reversal, it follows that 3 divides the reversal of every multiple of 3. That is, 3 is magic. Since a number is divisible by 9 if and only if the sum of its digits is divisible by 9, the same argument shows that 9 is magic.

A number is divisible by 11 if and only if the alternating sum of its digits is divisible by 11. For example, 1243 is divisible by 11 because $1-2+4-3=0$, which is divisible by 11. When the digits of a number are reversed, the alternating sum of digits remains the same, except for a possible change in sign. Therefore, 11 is magic.

A number is divisible by 33 if and only if it is divisible by both 3 and 11. Since reversal does not affect divisibility by 3 or 11, it also does not affect divisibility by 33. Therefore, 33 is magic. The same argument shows that 99 is magic.

2. Magic numbers are divisors of 99

Note that if n is divisible by 2 or 5, then n is not magic. To prove this fact, let m be the smallest multiple of n that has more digits than n does. The first digit of m must be 1, else m would be greater than twice the previous multiple $m-n$. Therefore, the reversal of m has 1 as its last digit. But no multiple of n can have 1 as its last digit, so n is not magic.

Let $n > 11$ be magic, and let $a = \phi(n)$. Note that a is even, and $10^a \equiv 1 \pmod{n}$ since n is coprime to 10. Let $b = (n-11)a/2$, which is a multiple

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of a since n is odd, and construct the following number:

$$N = 10^{b+1} + \sum_{k=0}^{n-11} 10^{ka} \ .$$

N is a multiple of n, since

$$N \equiv 10 + \sum_{k=0}^{n-11} 1 \equiv 0 \pmod{n} \ .$$

The reversal of N is

$$N' = 10^{b-1} + \sum_{k=0}^{n-11} 10^{ka} \ .$$

Since n is magic, N' is also a multiple of n . Therefore, the difference $N - N'$ is a multiple of n . But

$$N - N' = 10^{b+1} - 10^{b-1} = 99 \cdot 10^{b-1}.$$

Since n is coprime to 10, it follows that n divides 99.