# Unitarism and Infinitarism 

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We will examine variations of four famous arithmetical functions. For a given function $\chi$, let $\chi^{*}$ denote its unitary analog, $\tilde{\chi}$ its square-free analog, and $\chi^{\prime}$ its unitary square-free analog. The meanings of these phrases will be made clear in each case. At the end, the infinitary analog $\chi_{\infty}$ will appear as well.
0.1. Divisor Function. If $d(n)$ is the number of distinct divisors of $n$, then

$$
\sum_{n=1}^{N} d(n)=N \ln (N)+(2 \gamma-1) N+O(\sqrt{N})
$$

as $N \rightarrow \infty$, where $\gamma$ is the Euler-Mascheroni constant. Let us introduce a more refined notion of divisibility. A divisor $k$ of $n$ is unitary if $k$ and $n / k$ are coprime, that is, if $\operatorname{gcd}(k, n / k)=1$. This condition is often written as $k \| n$. The number $d^{*}(n)$ of unitary divisors of $n$ is $2^{\omega(n)}$, where $\omega(n)$ is the number of distinct prime factors of $n$. This fact is easily seen to be true: If $p_{1}^{a_{1}} p_{2}^{a_{2}} \cdots p_{r}^{a_{r}}$ is the prime factorization of $n$, then the unitary divisors of $n$ are of the form $p_{1}^{\varepsilon_{1} a_{1}} p_{2}^{\varepsilon_{2} a_{2}} \cdots p_{r}^{\varepsilon_{r} a_{r}}$, where each $\varepsilon_{s}$ is either 0 or 1 . There are $2^{r}$ possible choices for the $r$-tuple $\left(\varepsilon_{1}, \varepsilon_{2}, \ldots, \varepsilon_{r}\right)$; hence the result follows. We have $[1,2,3,4,5]$

$$
\sum_{n=1}^{N} d^{*}(n)=\frac{6}{\pi^{2}} N \ln (N)+\frac{6}{\pi^{2}}\left(2 \gamma-1-\frac{12}{\pi^{2}} \zeta^{\prime}(2)\right) N+O(\sqrt{N})
$$

where $\zeta(x)$ is the Riemann zeta function and $\zeta^{\prime}(x)$ is its derivative.
A divisor $k$ of $n$ is square-free if $k$ is divisible by no square exceeding 1 . The number $\tilde{d}(n)$ of square-free divisors of $n$ is also $2^{\omega(n)}$; the divisors in this case are of the form $p_{1}^{\varepsilon_{1}} p_{2}^{\varepsilon_{2}} \cdots p_{r}^{\varepsilon_{r}}$. Therefore the same asymptotics apply for $\tilde{d}(n)$, but the underlying sets of numbers overlap only somewhat [6].

Define $d^{\prime}(n)$ to be the number of unitary square-free divisors of $n$. A more complicated asymptotic formula arises here [7, 8]:

$$
\sum_{n=1}^{N} d^{\prime}(n)=\frac{6 \alpha}{\pi^{2}} N \ln (N)+\frac{6 \alpha}{\pi^{2}}\left(2 \gamma-1-\frac{12}{\pi^{2}} \zeta^{\prime}(2)+X\right) N+O(\sqrt{N} \ln (N))
$$

[^0]where
$$
\alpha=\prod_{p}\left(1-\frac{1}{p(p+1)}\right)=0.7044422009 \ldots, \quad X=\sum_{p} \frac{(2 p+1) \ln (p)}{(p+1)\left(p^{2}+p-1\right)}
$$
and we agree that the product and sum extend over all primes $p$. The constant $\alpha$ is the same as what is called $\pi^{2} P / 6$ in [9].

We finally give corresponding reciprocal sums [10, 11, 12]:

$$
\begin{gathered}
\lim _{N \rightarrow \infty} \frac{\sqrt{\ln (N)}}{N} \sum_{n=1}^{N} \frac{1}{d(n)}=\frac{1}{\sqrt{\pi}} \prod_{p} \sqrt{p(p-1)} \ln \left(\frac{p}{p-1}\right)=\frac{0.9692769438 \ldots}{\sqrt{\pi}} \\
\lim _{N \rightarrow \infty} \frac{\sqrt{\ln (N)}}{N} \sum_{n=1}^{N} \frac{1}{d^{*}(n)}=\frac{1}{\sqrt{\pi}} \prod_{p} \sqrt{1+\frac{1}{4 p(p-1)}}=\frac{1.0969831191 \ldots}{\sqrt{\pi}}
\end{gathered}
$$

The former sum was mentioned in [13] with regard to the arcsine law for random divisors. It is not known what constant emerges for $1 / d^{\prime}(n)$.
0.2. Sum-of-Divisors Function. If $\sigma(n)$ is the sum of all distinct divisors of $n$, then

$$
\sum_{n=1}^{N} \sigma(n)=\frac{\pi^{2}}{12} N^{2}+O(N \ln (N))
$$

as $N \rightarrow \infty$. Let $\sigma^{*}(n)$ be the sum of unitary divisors of $n$ and $\tilde{\sigma}(n)$ be the sum of square-free divisors of $n$. Although $d^{*}(n)=\tilde{d}(n)$ always, it is usually false that $\sigma^{*}(n)=\tilde{\sigma}(n)[14]$. We have $[15,16,17,18]$

$$
\lim _{N \rightarrow \infty} \frac{1}{N^{2}} \sum_{n=1}^{N} \sigma^{*}(n)=\frac{\pi^{2}}{12 \zeta(3)}, \quad \lim _{N \rightarrow \infty} \frac{1}{N^{2}} \sum_{n=1}^{N} \tilde{\sigma}(n)=\frac{1}{2}
$$

Further, if $\sigma^{\prime}(n)$ is the sum of unitary square-free divisors of $n$, then [15]

$$
\lim _{N \rightarrow \infty} \frac{1}{N^{2}} \sum_{n=1}^{N} \sigma^{\prime}(n)=\frac{1}{2} \prod_{p}\left(1-\frac{1}{p^{2}(p+1)}\right)=\frac{0.8815138397 \ldots}{2}
$$

a constant which appeared in [19] and turns out to be connected with class number theory [20, 21, 22].

Corresponding reciprocal sums are [23, 24]

$$
\sum_{n=1}^{N} \frac{1}{\sigma(n)} \sim Y_{1} \ln (N)+Y_{1}\left(\gamma+Y_{2}\right), \quad \sum_{n=1}^{N} \frac{1}{\sigma^{*}(n)} \sim Y_{3} \ln (N)+Y_{3}\left(\gamma+Y_{4}-Y_{5}\right)
$$

$$
\begin{aligned}
& \qquad Y_{1}=\prod_{p} f(p), \quad Y_{2}=\sum_{p} \frac{(p-1)^{2} g(p) \ln (p)}{p f(p)}, \\
& Y_{3}=\prod_{p}\left(1-\frac{p}{p-1} \sum_{j=1}^{\infty} \frac{1}{p^{j}\left(p^{j}+1\right)}\right), \quad Y_{4}=\sum_{p}\left(\frac{p h(p) \ln (p)}{p-1} \sum_{j=1}^{\infty} \frac{j}{p^{j}\left(p^{j+1}+1\right)}\right), \\
& Y_{5}=\sum_{p}\left(\frac{h(p) \ln (p)}{p^{2}} \sum_{j=0}^{\infty} \frac{1}{p^{j}\left(p^{j+1}+1\right)}\right), \quad f(p)=1-\frac{(p-1)^{2}}{p} \sum_{j=1}^{\infty} \frac{1}{\left(p^{j}-1\right)\left(p^{j+1}-1\right)}, \\
& g(p)=\sum_{j=1}^{\infty} \frac{j}{\left(p^{j}-1\right)\left(p^{j+1}-1\right)}, \quad h(p)=1-\frac{p}{p-1} \sum_{j=1}^{\infty} \frac{1}{p^{j}\left(p^{j+1}+1\right)} .
\end{aligned}
$$

No one seems to have examined $1 / \tilde{\sigma}(n)$ or $1 / \sigma^{\prime}(n)$ yet.
0.3. Totient Function. If $\varphi(n)$ is the number of positive integers $k \leq n$ satisfying $\operatorname{gcd}(k, n)=1$, then $[25,26]$

$$
\sum_{n=1}^{N} \varphi(n)=\frac{3}{\pi^{2}} N^{2}+O(N \ln (N))
$$

as $N \rightarrow \infty$. Define $\operatorname{gcd}_{*}(k, n)$ to be the greatest divisor of $k$ that is also a unitary divisor of $n$. Let $\varphi^{*}(n)$ be the number of positive integers $k \leq n$ satisfying $\operatorname{gcd}_{*}(k, n)=$ 1. Since $\operatorname{gcd}_{*}$ is never larger than gcd, it follows that $\varphi^{*}$ is at least as large as $\varphi$. Also let $\tilde{\varphi}(n)$ be the number of positive square-free integers $k \leq n$ satisfying $\operatorname{gcd}(k, n)=1$. We have [15, 27]

$$
\lim _{N \rightarrow \infty} \frac{1}{N^{2}} \sum_{n=1}^{N} \varphi^{*}(n)=\frac{1}{2} \alpha, \quad \lim _{N \rightarrow \infty} \frac{1}{N^{2}} \sum_{n=1}^{N} \tilde{\varphi}(n)=\frac{3}{\pi^{2}} \alpha
$$

where $\alpha$ is as defined earlier. The case for $\varphi^{\prime}(n)$ remains open.
Corresponding reciprocal sums are [23, 24, 28]

$$
\sum_{n=1}^{N} \frac{1}{\varphi(n)} \sim Z_{1} \ln (N)+Z_{1}\left(\gamma-Z_{2}\right), \quad \sum_{n=1}^{N} \frac{1}{\varphi^{*}(n)} \sim Z_{3} \ln (N)+Z_{3}\left(\gamma-Z_{4}+Z_{5}+Z_{6}\right)
$$

where

$$
Z_{1}=\frac{315 \zeta(3)}{2 \pi^{4}}, \quad Z_{2}=\sum_{p} \frac{\ln (p)}{p^{2}-p+1}, \quad Z_{3}=\prod_{p} u(p)
$$

$$
\begin{gathered}
Z_{4}=\sum_{p}\left(\frac{(p-1) \ln (p)}{p u(p)} \sum_{j=1}^{\infty} \frac{j}{p^{j}\left(p^{j}-1\right)}\right), \\
Z_{5}=\sum_{p} \frac{\ln (p)}{p^{2}(p-1) u(p)}, \quad Z_{6}=\sum_{p} \frac{v(p) \ln (p)}{p^{2} u(p)}, \\
u(p)=1+\frac{p-1}{p} \sum_{j=1}^{\infty} \frac{1}{p^{j}\left(p^{j}-1\right)}, \quad v(p)=\sum_{j=1}^{\infty} \frac{1}{p^{j}\left(p^{j+1}-1\right)} .
\end{gathered}
$$

0.4. Square-Free Core Function. If $\tilde{\kappa}(n)$ is the maximal square-free divisor of $n$ (also called [9] the square-free kernel of $n$ ), then $[15,17,18,29,30,31]$

$$
\sum_{n=1}^{N} \tilde{\kappa}(n)=\frac{\alpha}{2} N^{2}+O\left(N^{3 / 2}\right)
$$

as $N \rightarrow \infty$, where $\alpha$ is as before. Assuming the Riemann hypothesis, the error term can be improved to $O\left(N^{7 / 5+\varepsilon}\right)$ for any $\varepsilon>0$. If $\kappa^{\prime}(n)$ is the maximal unitary square-free divisor of $n$, then $[30,31]$

$$
\sum_{n=1}^{N} \kappa^{\prime}(n)=\frac{\beta}{2} N^{2}+O\left(N^{3 / 2}\right)
$$

where

$$
\beta=\prod_{p}\left(1-\frac{p^{2}+p-1}{p^{3}(p+1)}\right)=0.6496066993 \ldots
$$

0.5. Infinitary Arithmetic. We continue refining the notion of divisibility [32, 33]. A divisor $k$ of $n$ is biunitary if the greatest common unitary divisor of $k$ and $n / k$ is 1 , and triunitary if the greatest common biunitary divisor of $k$ and $n / k$ is 1 . More generally, for any positive integer $m$, a divisor $k$ of $n$ is $m$-ary if the greatest common $(m-1)$-ary divisor of $k$ and $n / k$ is 1 . We write $\left.k\right|_{m} n$. Clearly $\left.1\right|_{m} n$ and $\left.n\right|_{m} n$.

When introducing infinitary divisors, it is best to start with prime powers. Let $p$ be a prime, and let $x \geq 0, y \geq 1$ be integers. It can be proved that, for any $m \geq y-1$, $\left.p^{x}\right|_{m} p^{y}$ if and only if $\left.p^{x}\right|_{y-1} p^{y}$. Thus we define $\left.p^{x}\right|_{\infty} p^{y}$ if $\left.p^{x}\right|_{y-1} p^{y}$. For fixed $y$, the number of integers $0 \leq x \leq y$ satisfying $\left.p^{x}\right|_{\infty} p^{y}$ is $2^{b(y)}$, where $b(y)$ is the number of ones in the binary expansion of $y$. Define as well $\left.1\right|_{\infty} 1$. The sum $\sum_{y=0}^{z-1} 2^{b(y)}$ is approximately $z^{\ln (3) / \ln (2)}$ but is not well behaved asymptotically [34].

We now allow $n$ to be arbitrary. A divisor $k$ of $n$ is infinitary if, for any prime $p$, the conditions $p^{x}| | k$ and $p^{y}| | n$ imply that $\left.p^{x}\right|_{\infty} p^{y}$. We write $\left.k\right|_{\infty} n$. Clearly $\left.1\right|_{\infty} n$ and
$\left.n\right|_{\infty} n$. Each $n>1$ has a unique factorization as a product of distinct elements from the set

$$
I=\left\{p^{2^{j}}: p \text { is prime and } j \geq 0\right\}
$$

each element of $I$ in this product is called an $I$-component of $n$. It follows that $\left.k\right|_{\infty} n$ if and only if every $I$-component of $k$ is also an $I$-component of $n$.

Assume that $n=P_{1} P_{2} \cdots P_{t}$, where $P_{1}<P_{2}<\cdots<P_{t}$ are the $I$-components of $n$. The infinitary analogs of the functions $d$ and $\sigma$ are defined by [35, 36]

$$
d_{\infty}(n)=2^{t}, \quad \sigma_{\infty}(n)=\prod_{i=1}^{t}\left(P_{i}+1\right)
$$

for $n>1$; otherwise $d_{\infty}(1)=\sigma_{\infty}(1)=1$. Two infinitary analogs of the function $\varphi$ are known:

$$
\begin{gathered}
\varphi_{\infty}(n)=\text { the number of positive integers } k \leq n \text { satisfying } \operatorname{gcd}_{\infty}(k, n)=1 ; \\
\hat{\varphi}_{\infty}(n)=\prod_{i=1}^{t}\left(P_{i}-1\right)=n \prod_{i=1}^{t}\left(1-\frac{1}{P_{i}}\right) \quad \text { for } n>1, \quad \hat{\varphi}_{\infty}(1)=1
\end{gathered}
$$

It is generally untrue that $\varphi_{\infty}(n)=\hat{\varphi}_{\infty}(n)$. No similar extension of the function $\tilde{\kappa}$ is known. Cohen \& Hagis [35, 37] proved that

$$
\begin{gathered}
\lim _{N \rightarrow \infty} \frac{1}{N^{2}} \sum_{n=1}^{N} \sigma_{\infty}(n)=\frac{A}{2}=0.7307182421 \ldots, \\
\lim _{N \rightarrow \infty} \frac{1}{N^{2}} \sum_{n=1}^{N} \hat{\varphi}_{\infty}(n)=\frac{B}{2}=0.3289358388 \ldots, \\
\frac{1}{N^{2}} \sum_{n=1}^{N} d_{\infty}(n) \sim C N \ln (N)+D N \sim 2(0.3666252769 \ldots) N \ln (N)
\end{gathered}
$$

where

$$
A=\prod_{P \in I}\left(1+\frac{1}{P(P+1)}\right), \quad B=\prod_{P \in I}\left(1-\frac{1}{P(P+1)}\right), \quad C=\prod_{P \in I}\left(1-\frac{1}{(P+1)^{2}}\right)
$$

but no such expression for $D$ yet exists. It is known that $\varphi_{\infty}(n)=n^{2} / \sigma_{\infty}(n)+O\left(n^{\varepsilon}\right)$ for any $\varepsilon>0$; reciprocal sums involving $d_{\infty}, \sigma_{\infty}$ and $\hat{\varphi}_{\infty}$ also remain open. Alternative generalizations of unitary divisor have been given $[38,39]$ but won't be discussed here.
0.6. Addendum. The constant $X$, which is associated with counting unitary square-free divisors of $n$, was evaluated in [40] to be $0.7483723334 \ldots$. The series $\sum_{n \leq N} 1 / d(n)$ connected with ( $\left.0.5468559552 \ldots\right) N \ln (N)^{-1 / 2}$ was mentioned in [41, 42], as well as a numerical value $0.6728 \ldots$ for $Y_{1}$. Other values of $Y_{i}, Z_{j}$ remain open. An analog of $d^{\prime}(n)$, corresponding to unitary cube-free divisors of $n$, can be studied $[8,40]$. The probability that $k$ randomly chosen integers are unitary coprime is [43]

$$
\prod_{p}\left(1-\frac{(p-1)^{k}}{p^{k}\left(p^{k}-1\right)}\right)
$$

The probability that they are pairwise unitary coprime is more complicated: for instance, it is

$$
\zeta(2) \zeta(3) \prod_{p}\left(1-\frac{4}{p^{2}}+\frac{7}{p^{3}}-\frac{9}{p^{4}}+\frac{8}{p^{5}}-\frac{2}{p^{6}}-\frac{3}{p^{7}}+\frac{2}{p^{8}}\right)
$$

when $k=3$ and

$$
\begin{array}{r}
\zeta(2)^{2} \zeta(3) \zeta(4) \prod_{p}\left(1-\frac{8}{p^{2}}+\frac{3}{p^{3}}+\frac{27}{p^{4}}-\frac{24}{p^{5}}-\frac{14}{p^{6}}-\frac{3}{p^{7}}\right. \\
\left.\quad+\frac{37}{p^{8}}-\frac{30}{p^{9}}+\frac{42}{p^{10}}-\frac{33}{p^{11}}-\frac{41}{p^{12}}+\frac{78}{p^{13}}-\frac{44}{p^{14}}+\frac{9}{p^{15}}\right)
\end{array}
$$

when $k=4$. Expressions for arbitrary $k$ appear in [43].

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