Scan 7629
Keith

JRM v18n4 p 275-8, 1986.

POWER-SUM NUMBERS

MICHAEL KEITH
115 Oakdale Avenue
Washington Crossing, Pennsylvania 18977

Introduction

Consider integers of the following type:

$$336 = (3 + 3 + 6) + (3^2 + 3^2 + 6^2) + (3^3 + 3^3 + 6^3)$$
$$21 = (2 + 1) + (2^3 + 1^3) + (2^3 + 1^3)$$

that is, integers equal to a sum of sums of powers of their digits. This generalizes the notion of numbers equal to sums of powers of their digits (such as 153, which is equal to the sum of the cubes of its digits). In words, we can say that 336 is equal to the sum of its digits plus the squares of its digits plus the cubes of its digits.

More precisely, we have the following definition:

Definition: Let N be an integer with k digits. Let

$$S_{e_1, e_2, \ldots, e_m}(N) = \sum_{j=1}^{k} \sum_{i=1}^{m} d_j^{e_i}$$

where the d_j are the digits of N. This function (we usually just write $S(N)$, omitting the subscripts) is called the *power sum* of N. If $S(N) = N$, we say that N is a *power-sum number* of class $e_1, \ldots, e_m$.

$PS(e_1, \ldots, e_m)$ denotes the set of all power-sum *(PS)* numbers of class $e_1, \ldots, e_m$. A PS number with all the e_i distinct is called a *proper* power-sum *(PPS)* number, and the corresponding sets are denoted $PPS(e_1, \ldots, e_m)$.

Stated in this terminology, the two examples above show us that

$$336 \in PS(1,2,3) \quad \text{and} \quad 21 \in PS(1,3,3)\,.$$

Note that 336 is also proper since the exponents are all distinct.

#7629

The question which arises, of course, is to find all the members of a particular *PS* set. We immediately have the following theorem.

Theorem: Every *PS* set is a finite set.

Proof: Consider all the integers N with a fixed number of digits k. The smallest N can be is 10^{k-1}. The largest $S(N)$ can be is for a number consisting of all 9's; for this value of N,

$$S(N) = k \sum_{i=1}^{m} 9^{e_i} .$$

Now, compare this largest value of $S(N)$ with the smallest value of N. We see that as k increases, the smallest N grows exponentially, while the largest $S(N)$ grows only linearly. Thus, for some constant M, we will have

$$10^{k-1} > k \sum_{i=1}^{m} 9^{e_i} \quad (\text{all } k \geqslant M) . \qquad (1)$$

Since a *PS* number requires that these two expressions be equal, this means that no *PS* numbers exist for $k \geqslant M$.

This theorem is also useful in that it provides an upper bound in searching for *PS* numbers. We need only find the value of M (the smallest value of k such that inequality 1 holds) and then search only the integers in the range 1 to $S(10^{M-1} - 1)$. For example, for $PS(1,3)$ we find that $M = 5$ so we only have to search up to $S(9999) = 2952$.

Numerical Results

Table 1 shows the result of searching for *PS* numbers for the first few sets of small exponents. This table covers all sets with either $m \leqslant 2$ and $k \leqslant 4$ or with $m \leqslant 4$ and $k \leqslant 3$. Here are a few observations and questions gleaned from study of this table.

1. The largest *PS* sets so far (other than the trivial $PS(1)$) are $PS(1,3)$ and $PS(1,1,3,3)$, both with 6 members. The smallest sets are the several empty sets.
2. Can the empty sets be characterized? Are there an infinite number of empty sets?
3. There are an infinite number of non-empty *PS* sets, since, for example, $50\ldots0$ (t digits) is a member of $PS(2,2,\ldots,2)$ ($10t$ 2's). But are there an infinite number of *proper* sets?
4. Note that if m is even, all members of all *PS* sets for that m are also even. This is easily seen to be true in general.

Table 1. *PS* Numbers for Small Exponents

$e_1e_2 \ldots e_m$	Members of $PS\ (e_1, e_2, \ldots, e_m)$
1	1 2 3 4 5 6 7 8 9
2	1
3	1 153 370 371 407
4	1 1634 8208 9474
11	18
12	90
13	12 30 666 870 960 1998
14	7816
22	50 298
23	—
24	20
33	702 1728
34	3174 5756
44	14758
111	27
112	23 63 80
113	717 954
122	—
123	336
133	21 315 1926
222	267
223	—
233	20 550
333	—
1111	12 24 36 48
1112	70
1113	262 346 674 846
1122	40
1123	134 306 670 1096
1133	20 140 700 720 1184 2992
1222	30 172 218 228 396
1223	104 636 900
1233	444 2688
1333	—
2222	376
2223	20 244
2233	1944 2488
2333	—
3333	500 2080 2376 2580 3784

Finally, we ask, are there any numbers which are not *PS* numbers of any class? The answer is yes: 11 is the smallest such number. Since all its power-sums equal 2, they can never add to 11. In fact, there are many *non-PS* numbers; the first few are:

11,13,14,15,16,17,19,22,25,26,28,29,31,32,33,34,35,37,38,39, ...

There are clearly an infinite number of these as well since, for example, all repunits with an even number of digits are non-*PS* numbers.