TWIN PRIME CONJECTURES

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1. Introduction

Goldbach's famous conjecture, that every even integer greater than 2 is the sum of two primes, is over 250 years old. It is indeed an intriguing problem since it is so easy to state and understand, yet its proof has resisted the efforts of some of the finest mathematicians of all time. An excellent summary of the current situation is contained in Paulo Ribenboim's book [1].

In October , 1996, Ribenboim asked me to investigate a Goldbach-like conjecture about twin primes that was suggested to him by Stephen Wagler of California. Define a"middle number" to be m, the number sandwiched between a pair of twin primes $m-1$ and $m+1$. A t-prime is defined as a prime which has a twin. I was asked to search for the smallest middle number which is not the sum of two t-primes. I initially misinterpreted the request and instead searched for middle numbers which were not the sum of two middle numbers but none were found. The process of correcting this mistake suggested several related conjectures and resulted in developing interesting data about twin primes. The only previously published work on the subject that was found was a one-page paper by Zwillinger [2].

2. CONJECTURES

Conjecture 1: *Every middle number greater than 6 is the sum of two middle numbers.*
Conjecture 2: *Every middle number greater than 6 is the sum of two t-primes.*
Conjecture 3: *Every sufficiently large number divisible by 6 is the sum of two middle numbers.*
Conjecture 4: *Every sufficiently large even number is the sum of two t-primes.*

The analogy to Goldbach's conjecture is clear. It is interesting and somewhat unexpected that essentially the same Goldbach conditions apply to the much less dense twin primes.

Note that every middle number greater than 4 is a multiple of 6 so that for sufficiently large numbers, Conjecture 1 is included in Conjecture 3. If

$$\begin{aligned} N &= m_1 + m_2, && \text{then} \\ N &= (m_1+1)+(m_2-1), && \\ N+2 &= (m_1+1)+(m_2+1), && \textit{and} \\ N-2 &= (m_1-1)+(m_2-1). && \end{aligned}$$

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The left hand sides include all the even numbers, which means that Conjecture 3 implies Conjecture 4, Conjecture 1 implies Conjecture 2 and conversely.

The original objective of this project was to find middle numbers that were not the sum of two middle numbers since the relative scarcity of twin primes seemed to assure their existence. However, after some initial searching it became more and more likely that such numbers may not exist, hence Conjecture 1. More searching suggested replacing Conjectures 3 and 4 with

> **Conjecture 3a:** *Every number divisible by six which is greater than 4206 is the sum of two middle numbers.*
> **Conjecture 4a:** *Every even number greater than 4208 is the sum of two t-primes.*

All the above conjectures are closely related, and in fact Conjecture 3a (with a little extra data) implies all the others.

As N increases so does the number of ways that it is the sum of two middle numbers (on average). For example, 2,840,448 (which happens to be the middle number for the 10,000th twin) is the sum of two middle numbers in 528 ways, with the smallest middle number included in a sum being 30. However as N increases there are large variations in the number of sums and of the smallest middle number. It seemed reasonable to investigate these variations in an attempt to find data that would tend to support or refute the conjectures.

3. Search Results

Let N be a positive integer divisible by 6 and greater than 6, with $m_1,\ m_2$ middle numbers greater than 4. It is obvious from the characteristics of twin primes that all such m are also divisible by 6. Consider the equation,

$$N = m_1 + m_2, \quad m_1 \leq m_2 \ .$$

By direct search we found that there was no solution for the following values of N:

96, 402, 516, 786, 906, 1116, 1146, 1266, 1356, 3246, 4206.

There was at least one solution for all other N up to 20,000,000,000. None of the above values of N are middle numbers so that there are no middle numbers in the search range which contradict Conjecture 1. The largest N without a solution is 4206, hence Conjecture 3a.

In a preliminary search, samples were tabulated of the number of ways that a number was the sum of two middle numbers as well as the smallest middle number included in a sum. These samples are shown in Table 1. Examining them gives an indication that N's which have a high least m_1 tend to have fewer total hits. Although this is not a very tight correlation it seemed reasonable to investigate this in detail. While checking to see that each N satisfied Conjecture 3a, maximal m_1's were recorded. That is, a maximal m_1 exceeds all previous m_1's. After searching up to 2.10^{10}, the values of N which produced a maximal m_1 were examined to determine the number of ways that they were the sum of two middle numbers (hits). Also, for purposes of comparison the number of ways such N satisfied Goldbach's conjecture was also determined.

These results are shown in Table 2. Although the maximal smallest m_1 becomes quite large, the number of hits keeps increasing until it is intuitively obvious that the hits will never become zero. To see if any further insights could be obtained we searched for twin prime maximal gaps up to 4.10^{11}. They are tabulated in Table 3. These maximal gaps are considerably smaller than the maximal smallest m_1's in Table 2. Thus it seems unlikely that there are anomalies in the twin gap statistics that will have a serious effect on the conjectures.

It is not reasonable to expect these conjectures to be proved true in the near future. Their truth would imply an infinite number of twin primes as well as the truth of the Goldbach conjecture.

References

1. D. Zwillinger, *A Goldbach Conjecture Using Twin Primes*, Math. Comp. 33, July, 1979, p 1071.
2. P. Ribenboim, *The New Book of Prime Number Records*, Springer-Verlag, 1996, pp 291-299.

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Table 1. Samples of Least m_1 and Total Hits. $N = m_1 + m_2$

N	type	least m_1	total hits	N	type	least m_1	total hits
12	TWIN	6	1	100002	even	12	64
18	TWIN	6	1	100008	even	18	37
24	even	6	2	100014	even	882	32
30	TWIN	12	1	100020	even	30	69
36	even	6	2	100026	even	2238	17
42	TWIN	12	1	100032	even	42	59
48	even	6	2	100038	even	1230	32
54	even	12	1	100044	even	1482	30
60	TWIN	18	2	100050	even	60	72
66	even	6	1	100056	even	348	42
72	TWIN	12	2	100062	even	72	43
78	even	6	2	100068	even	348	33
84	even	12	2	100074	even	3252	23
90	even	18	2	100080	even	822	54
96	even	18	0	100086	even	828	30
102	TWIN	30	2	100092	even	102	41
108	TWIN	6	1	100098	even	108	70
114	even	6	3	100104	even	2802	18
120	even	12	3	100110	even	1302	63
126	even	18	1	100116	even	858	24
1002	even	180	3	1000002	even	42	167
1008	even	150	4	1000008	even	1320	339
1014	even	192	1	1000014	even	4422	84
1020	TWIN	138	4	1000020	even	60	413
1026	even	6	2	1000026	even	1488	83
1032	TWIN	12	4	1000032	even	72	277
1038	even	6	5	1000038	TWIN	2730	205
1044	even	12	2	1000044	even	6	91
1050	TWIN	18	6	1000050	even	12	328
1056	even	6	3	1000056	even	18	96
1062	TWIN	12	7	1000062	even	102	282
1068	even	6	3	1000068	even	30	164
1074	even	12	4	1000074	even	462	130
1080	even	18	7	1000080	even	42	248
1086	even	228	1	1000086	even	3918	84
1092	TWIN	30	8	1000092	even	660	239
1098	even	6	3	1000098	even	60	211
1104	even	12	5	1000104	even	1452	125
1110	even	18	4	1000110	even	72	276
1116	even	18	0	1000116	even	1428	127
10002	even	72	10	10000002	even	30	1259
10008	TWIN	150	13	10000008	even	2130	796
10014	even	6	7	10000014	even	42	645
10020	even	12	19	10000020	even	858	1710
10026	even	18	7	10000026	even	14628	406
10032	even	102	17	10000032	even	60	1717
10038	TWIN	30	19	10000038	even	108	827
10044	even	6	5	10000044	even	72	770
10050	even	12	18	10000050	even	1932	1277
10056	even	18	6	10000056	even	3918	506
10062	even	432	16	10000062	even	1092	1151
10068	TWIN	30	14	10000068	even	138	1042
10074	even	6	6	10000074	even	102	703
10080	even	12	24	10000080	even	108	2020
10086	even	18	7	10000086	even	4518	540
10092	TWIN	462	17	10000092	even	1950	953
10098	even	6	18	10000098	even	1050	967
10104	even	12	6	10000104	even	6762	620
10110	even	18	21	10000110	even	138	1270
10116	even	108	10	10000116	even	1998	802

Table 2. Maximal Smallest Middle Numbers up to 2.10^{10}

$N = m_1 + m_2, m_1 \leq m_2, 6|N$

maximal smallest m_1	N	total hits	total hits Goldbach
6	12	1	1
12	30	1	3
18	60	2	6
30	102	2	8
72	174	1	11
108	306	1	15
198	396	1	21
348	696	1	30
570	1998	1	56
828	2526	2	68
858	2946	2	87
1278	3156	2	85
1428	3216	2	87
1608	3846	1	98
1668	5226	2	131
1872	6354	4	140
2088	6606	3	138
3168	7386	3	156
3258	8046	3	169
3528	10986	3	219
4128	11586	2	218
4338	12876	3	249
5022	17634	3	305
11718	24096	1	399
12162	62754	11	860
13338	76926	9	973
16902	100134	12	1230
17988	200166	20	2148
23688	260166	24	2668
25998	519396	51	4670
31512	640104	52	5683
41232	953754	72	7841
43578	2454096	184	19000
56208	3807126	222	25402
59358	4887186	232	31432
76158	13122546	521	73913
95232	15477414	629	85399
117702	64752654	1634	297604*
129918	146511996	3269*	624712*
146382	195116874	4043*	798607*
151008	296060526	5361*	1146617*
153522	309376854	5505*	1192279*
195342	392287284	6780*	1475374*
203382	696090114	10575*	2461094*
230862	1169611134	17569*	4132695*
232752	2277008544	28942*	
236892	3170423574		
241782	4644147864		
250968	5382273786		
265542	5624850204		
271278	5965008906		
289242	6347148504		
299358	8115525846		
312198	10000538376		
375708	16963308576		

Note: * means that some numbers are called prime based on a Fermat test only so that numbers are not exact.

Table 3. Twin Prime Maximal Gaps up to $4 \cdot 10^{11}$

maximal gap	low middle number
2	4
6	6
12	18
18	42
30	72
36	312
72	348
150	660
168	2382
210	5880
282	13398
372	18540
498	24420
630	62298
924	187908
930	687522
1008	688452
1452	850350
1512	2868960
1530	4869912
1722	9923988
1902	14656518
2190	17382480
2256	30752232
2832	32822370
2868	96894042
3012	136283430
3102	234966930
3180	248641038
3480	255949950
3804	390817728
4770	698542488
5292	2466641070
6030	4289385522
6282	19181736270
6474	24215097498
6552	24857578818
6648	40253418060
7050	42441715488
7980	43725662622
8040	65095731750
8994	134037421668
9312	198311685750
9318	223093059732
10200	353503437240