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CLASSIFYING AND COUNTING HEXAFLEXAGRAMS

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The articles by Martin Gardner [1, 2] were my first contact with hexaflexagrams. The history of their discovery and how a theory of their construction and operation was developed by four graduate students at Princeton University in 1939 is recounted.

A structure diagram for each possible hexaflexagram is given in the book by Madachy [3]. The structure diagram for a J-sided hexaflexagram is constructed by completely triangulating J vertices, where similar figures that result are considered equivalent. The J vertices of the resulting diagram represent the J sides of the hexaflexagram and each connecting edge represents one of the possible states. The end points of each edge represent the external sides for that particular state. Figure 1 shows the possible state diagrams for $J = 3, 4, 5, 6$. $J = 6$ is the lowest number for which there can be more than one possible hexaflexagram.

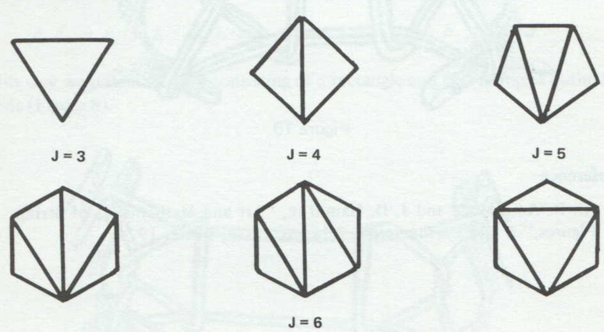

Figure 1

Counting the number of possible J-sided hexaflexagrams is equivalent to determining the number of distinct J-point structure diagrams that can be constructed. The diagrams can be built up systematically if it is noted that a $(J + 1)$-point diagram results from a J-point diagram by externally inserting a point and a two-line triangular connection between two points of a J-point diagram. In each J-point diagram there are J external edges and there are J possible $J + 1$ resulting diagrams. This is shown in Figure 2, which gives all the five possible 6-point diagrams that result from expanding the 5-point diagram. Examination of the result shows that there are but three distinct patterns and these are the ones shown in Figure 1.

The resulting three 6-point figures could be expanded giving eighteen possible 7-point figures, but examination of these would show that only four of them are distinct. This process could then be repeated to generate the possible 8-point diagrams and then these could be examined for uniqueness. The tedium of such a method is obvious, but the fact that there is a definite routine suggests that a computer may be called on for assistance. To do this, it is first necessary to devise a numerical representation for each structure diagram. This is done by representing each J-point diagram by an order J-tuple of digits, where each digit represents a vertex and is equal to the number of edges that terminate in that vertex. The J-tuple is ordered by arranging the vertex digits in the order that they occur in the diagram and choosing the first digit and succeeding digits such that the largest number equivalent to the Arabic-Hindu positional notation results. The J-tuples for $J = 3$ through 8 are listed in Table 1 and for J up to 6 these can be checked with Figure 1. In the table J is the number of points in a diagram and K is the counter for the number of J-point diagrams. Increasing K corresponds to decrease in the J-tuples with respect to the Arabic-Hindu notation. The notation that is adopted is that $H(J, K)$ stands for an ordered J-tuple, $H(J)$ stands for a complete set of possible J-tuples and $N(J)$ stands for the number of elements in a set. An example of a number and of a set are given below.

$$H(7, 3) = (5, 2, 4, 2, 4, 3, 2)$$
$$H(7) = (H(7, 1), H(7, 2), H(7, 3), H(7, 4))$$

If the set of J-tuples $H(J)$ are kept in descending order, the procedure for generating the $H(J + 1)$ set can be specified and simplified. The basic procedure is to expand an $H(J, K)$ number into a $(J + 1)$-tuple. This is done by inserting a 2 into any of the J-digit positions of $H(J, K)$. The position into which the 2 is inserted will be designated as position I. All of the digits in the positions I or greater are shifted to the next greater position and the digits now in positions $I - 1$ and $I + 1$ are increased by one. The results is a $(J + 1)$-tuple which may have to be rearranged to be put into the standard form. This procedure is equivalent to expanding a structure diagram by adding a triangle as shown in Figure 2. The expansion of $H(J,K)$ by inserting a 2 in digit position I will be

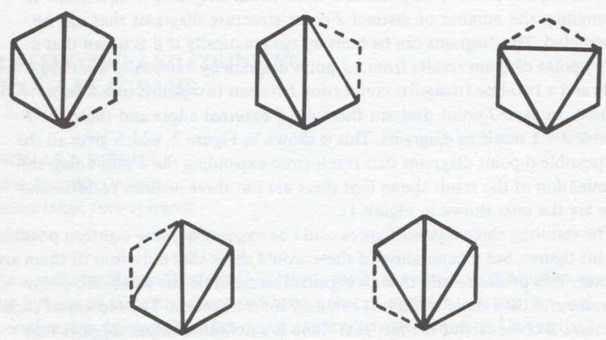

Figure 2

Table 1.

J	*K*	*H(J, K)*
3	1	(2, 2, 2)
4	1	(3, 2, 3, 2)
5	1	(4, 2, 3, 3, 2)
6	1	(5, 2, 3, 3, 3, 2)
	2	(4, 3, 2, 4, 3, 2)
	3	(4, 2, 4, 2, 4, 2)
7	1	(6, 2, 3, 3, 3, 3, 2)
	2	(5, 3, 2, 4, 3, 3, 2)
	3	(5, 2, 4, 2, 4, 3, 2)
	4	(4, 4, 2, 3, 4, 3, 2)
8	1	(7, 2, 3, 3, 3, 3, 3, 2)
	2	(6, 3, 2, 4, 3, 3, 3, 2)
	3	(6, 2, 4, 2, 4, 3, 3, 2)
	4	(6, 2, 3, 4, 2, 4, 3, 2)
	5	(5, 4, 2, 3, 4, 3, 3, 2)
	6	(5, 3, 3, 2, 5, 3, 3, 2)
	7	(5, 3, 2, 5, 2, 4, 3, 2)
	8	(5, 3, 2, 4, 4, 2, 4, 2)
	9	(5, 3, 2, 4, 3, 4, 2, 3)
	10	(5, 2, 5, 2, 3, 4, 3, 2)
	11	(5, 2, 4, 2, 5, 2, 4, 2)
	12	(4, 4, 3, 2, 4, 4, 3, 2)

designated by EX(I, $H(J, K)$). In the examples that follow, the first results in a standard form immediately, the second has to be shifted to the left and the third has to be reversed and shifted.

$$\mathrm{EX}(3, H(7,2)) = (5,4,2,3,4,3,3,2)$$
$$\mathrm{EX}(5, H(7,3)) = (5,2,4,3,2,5,3,2) = (5,3,2,5,2,4,3,2)$$
$$\mathrm{EX}(6, H(7,4)) = (4,4,2,3,5,2,3,2) = (5,3,2,4,4,2,3,2)$$

If all of the Jx$N(J)$ digits of the $H(J)$ set are expanded, the digits rearranged if necessary, and the unique numbers retained, this will give the $H(J + 1)$ set. However, this procedure can be simplified before the final algorithm is stated.

If in the expansion of a number in the set $H(J)$ there is a digit to the right of the expanding digit, excluding the Jth digit, then the resulting expanded number could have been realized by expanding a number of $H(J)$ which is greater in magnitude. Expanding by inserting a 2 digit and increasing the neighboring digits gives a larger sequence for the digits to the left of the inserted 2 digit. If there is a 2 digit to the right, the $(J + 1)$-tuple could have resulted by expanding this digit in the number which had the larger sequence. The Jth digit is not counted as being to the right because the numbers are really cyclical and the Jth digit is a neighbor of the first digit. An example is as follows:

$$\mathrm{EX}(2, H(7,3)) = (6,2,3,4,2,4,3,2) = \mathrm{EX}(5, H(7,1))$$

and

$$H(7,1) > H(7,3)$$

This result suggests that not all of the possibilities of expansion need to be examined. In the expansion of each number, the expansion digits need only be chosen starting at the position of the rightmost 2 digit and increasing to the Jth digit. Then the $(J+1)$-tuples that result are generated in decreasing order. If the numbers to be expanded are chosen in descending order then all the $(J + 1)$-tuples that result from expanding the set $H(J)$ are generated in descending order.

Some of the $(J + 1)$-tuples will have to be rearranged in order to be put into the standard form. Any rearrangement that is necessary means that a larger number can be made from a particular sequence, but the procedure that results in $(J + 1)$-tuples in decreasing order implies that such a number is a duplicate of a previously generated larger number. When all of the numbers that have to be rearranged are discarded, then the $(J + 1)$-tuples that remain are the set $H(J + 1)$ in descending order.

Starting with the numbers of the set $H(J)$ arranged in descending order, and following the procedure described, gives the set $H(J + 1)$ with the numbers arranged in descending order. The procedure is shown in Table 2 which gives the

Table 2.

(5, 2, 3, 3, 3, 2)	
(6, 2, 3, 3, 3, 3, 2)	
(5, 3, 2, 4, 3, 3, 2)	
(5, 2, 4, 2, 4, 3, 2)	
(5, 2, 3, 4, 2, 4, 2)	(5, 2, 4, 2, 4, 3, 2)
(5, 2, 3, 3, 4, 2, 3)	(5, 3, 2, 4, 3, 3, 2)
(4, 3, 2, 4, 3, 2)	
(4, 4, 2, 3, 4, 3, 2)	
(4, 3, 3, 2, 5, 3, 2)	(5, 3, 2, 4, 3, 3, 2)
(4, 3, 2, 5, 2, 4, 2)	(5, 2, 4, 2, 4, 3, 2)
(4, 3, 2, 4, 4, 2, 3)	(4, 4, 2, 3, 4, 3, 2)
(4, 2, 4, 2, 4, 2)	
(4, 2, 5, 2, 3, 4, 2)	(5, 2, 4, 2, 4, 3, 2)
(4, 2, 4, 3, 2, 5, 2)	(5, 2, 4, 2, 4, 3, 2)
(4, 2, 4, 2, 5, 2, 3)	(5, 2, 4, 2, 4, 3, 2)

results of expanding the three numbers of the set $H(6)$. There are twelve 7-tuples that result, but only four of them do not have to be rearranged, and these four make up the set $H(7)$. Starting with the initial value (2, 2, 2), which is the only number in the set $H(3)$, all of the succeeding sets can be generated and counted.

The results of the count of $N(J)$, up to $J = 13$, are listed in Table 3 and agree with the calculations given by Oakley and Wisner [4]. Listed in the third column is the actual number of J-tuples, $A(J)$, that had to be generated to produce the next set. The fourth column gives $A(J)/(J\text{x}N(J))$, which is the ratio of the actual number of J-tuples generated to the total number that would result if all of the digits of all the numbers of the set $H(J)$ were expanded. This is a figure of merit that indicates that the algorithm becomes relatively more efficient as J increases.

Table 3.

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J	$N(J)$	$A(J)$	$\frac{A(J)}{J\text{x}N(J)}$
3	1	2	0.67
4	1	3	0.75
5	1	4	0.80
6	3	12	0.67
7	4	20	0.71
8	12	55	0.57
9	27	127	0.52
10	82	371	0.45
11	228	1037	0.41
12	733	3249	0.36
13	2282		

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Thomas J. O.Reilly is an Associate Professor of Mathematics at Saint Joseph's College in Philadelphia, Pennsylvania. He had previously worked for Burroughs Corporation, Philco Corporation, and the Frankford Arsenal. He has a B.S. from Boston College and an M.A. and Ph.D from Temple University. His fields of research interest have included solid state current flow, ligand field theory, numerical analysis, and switching theory.