The Am. Math. Mo. v100 n3 p 233 March 1993 A7459

**Theorem 3.1.** *The finite quotients of* **N** *are the* $\mathbf{N}_{a,k}$ *where* $a, k \in N$ *satisfy (for all primes* $p$*):*

$$p^e | k \Rightarrow e \le a$$

$$p | k \Rightarrow (p-1) | k.$$

Let $\mathbf{N}_{a,k}$ be a quotient of **N** with $k > 1$, and let $k = p_1^{e_1} \cdots p_r^{e_r}$ with $p_1 < \cdots < p_r$. As $p_1 - 1 | k$ and all the prime factors of $k$ are larger than $p_1 - 1$ it follows that $p_1 - 1 = 1$; thus $p_1 = 2$. And if $r > 1$ then $p_2 - 1 | k$ leads to $p_2 - 1 | p_1^{e_1}$, so $p_2$ is of the form $2^m + 1$, and thus it is a Fermat prime. The next corollary gives a complete list of the five "circle" integer HSI-algebras, i.e., those with $a = 1$, and hence no "tail."

**Corollary 3.2.** [D. Higgs]. $\mathbf{N}_{1,k}$ *is a quotient of* **N** *iff* $k \in \{1, 2, 6, 42, 1806\}$.

Jeff Shallit pointed out that the related sequence 2, 3, 7, 43, 1807 occurs in a number of places in the literature, e.g., as solutions to Sylvester's recurrence equations (see Davison & Shallit [3], Sylvester [8], [9]).

Given $a \in N$ define the sequence of primes $\Sigma_a = (p_1, p_2, \cdots)$ by

- $p_1 = 2$;
- given $p_1, \ldots, p_i$, let $p_{i+1}$ be the smallest prime $p$ which is greater than $p_i$ and such that $(p-1) | (p_1 \cdots p_i)^a$, assuming such a $p$ exists. If no such $p$ exists then $\Sigma_a$ terminates with $p_i$.

**Proposition 3.3.** *Given a positive integer* $a$, *there are infinitely many* $\mathbf{N}_{a,k}$ *iff the sequence of primes* $\Sigma_a$ *is infinite.* $\Pi_a$ might be a better symbol.

By Higgs's result $\Sigma_1 = (2, 3, 7, 43)$, a finite sequence. However it is not known if $\Sigma_a$ is finite for all (any) $a > 1$. We found that about 20% of the primes below 1,000,000 are in $\Sigma_2 = (2, 3, 5, 7, 11, 13, 19, 23, \ldots, 999667, 999727, \ldots)$—so even if $\Sigma_2$ is finite, a computer enumeration does not look feasible.

**4. SOME CONSTRUCTIONS OF HSI-ALGEBRAS.** We have seen one easy way to construct finite HSI-algebras, by taking quotients of **N**. Next we give five ways to construct finite HSI-algebras which have apparently nothing to do with **N**. In four of the cases below we let exponentiation be the *first projection* function $\pi$ (defined by $\pi(a, b) = a$).

- Let $\mathbf{H} = \langle H, \vee, \wedge, \rightarrow, 0, 1 \rangle$ be a Heyting algebra. Then $\mathbf{H}^* = \langle H, \vee, \wedge, \leftarrow, 1 \rangle$ is an HSI-algebra, where $a \leftarrow b$ is defined to be $b \rightarrow a$.
- Let $\mathbf{D} = \langle D, \vee, \wedge, 1 \rangle$ be a distributive lattice with 1. Then $\langle D, \vee, \wedge, \pi, 1 \rangle$ is an HSI-algebra.
- Let $\mathbf{S} = \langle S, \wedge, 1 \rangle$ be a semilattice with 1. Then $\langle S, \wedge, \wedge, \pi, 1 \rangle$ is an HSI-algebra.
- Let $\mathbf{S} = \langle S, \wedge, 0, 1 \rangle$ be a semilattice with 0, 1. Then $\langle S, f, \wedge, \pi, 1 \rangle$ is an HSI-algebra, where $f$ is the binary constant map whose value is always 0.
- Let $\mathbf{R} = \langle R, +, \times, 0, 1 \rangle$ be a Boolean ring. Then $\langle R, +, \times, \pi, 1 \rangle$ is an HSI-algebra.

« Drop NextP Dup Factor »

M∅642

~~2 · 3 · 5 · 7  11 · 13 · 19 · 23  29 · 31 · 37 · 43~~

~~47 · 53 · 59 · 61  67 · 71 · 79 · 83  101 · 107 · 131 · 139~~

2 · 3 · 5 · 7  11 · 13 · 19 · 23  29 · 31 · 37 · 43

47 · 53 · 59 · 61  67 · 71 · 79 · 101  107 · 127 · 131 · 139

149 · 151 · 157 · 173  181 · 191 · 197 · 199  211 · 223 · 229 · 263

269 · 277 · 283 · 311  317 · 331 · 347 · 349  367 · 373 · 383 · 397

419 · 421 · 431 · 461  463 · 491 · 509 · 523  547 · 557 · 571 · 599

607 · 631 · 643 · 659  661 · 677 · 683 · 691  701 · 709 · 727 · 733

743 · 787 · 797 · 827  829 · 839 · 853 · 859  863 · 883 · 907 · 911

941 · 947 · 967 · 983  991 · 1013 · 1019 · 1039  1051 · 1061 · 1063 · 1087

1093 · 1103 · 1109 · 1117 · 1151

Neil, the factors of 83-1 or 82 = 2 * 41. 41 is not on the list so ∴ 83 cannot be on the list

**M064**[illegible] 2,3,5,7,11,13,17,31,37,71,73,[illegible],97,101,107,113,131,149,151,[illegible],167,
179,181,191,199,311,313,337,347,353,359,373,383,389,701,709,727,733,739
Palindromic primes: reversal is prime. Ref rgw. [1,1; A7500]

**M0641** 2,3,5,7,11,13,17,31,37,71,73,79,97,113,131,199,311,337,373,733,919,
991,1111111111111111111,11111111111111111111111
Every permutation of digits is a prime. Ref MMAG 47 233 74. rcs. [1,1; A3459]

**M0642** 2,3,5,7,11,13,19,23,29,31,37,43,47,53,59,61,67,71,79,~~83~~,101
Higgs' primes: $a(n+1)$ = next prime such that $a(n+1)-1 \mid (a(1)...a(n))^2$. Ref AMM 100 233 93. [1,1; A7459]

**M0643** 1,2,3,5,7,11,13,19,23,29,33,43,47,59,65,73,81,97,103,121,129,141,
151,173,181,201,213,231,243,271,279,309
Fractions in Farey series of order $n$. Ref AMM 95 699 88. [1,2; A5728]

**M0644** 1,1,2,3,5,7,11,14,20,26,35,44,58,71,90,110,136,163,199,235,282,331,
391,454,532,612,709,811,931,1057,1206,1360,1540,1729,1945,2172,2432
Partitions of $n$ into at most 6 parts. Ref CAY 10 415. RS4 2. [0,3; A1402, N0243]

**M0645** 1,1,2,3,5,7,11,15,22,30,42,56,77,101,135,176,231,297,385,490,627,
792,1002,1255,1575,1958,2436,3010,3718,4565,5604,6842,8349,10143,12310
Partitions of $n$. See Fig M0645. Ref RS4 90. R1 122. AS1 836. [0,3; A0041, N0244]

$$\text{G.f.: } \prod_{n=1}^{\infty} (1 - x^n)^{-1}.$$

Fig M

**M0646** 1,2,3,5,7,11,16,26,40,65,101,163,257,416,663,1073,1719,2781,4472,
7236,11664,18873,30465,49293,79641,128862,208315,337061,545071