Proof of a conjecture stated in A007405

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For $n \in \mathbb{N}_0$ let $a(n) = $ A007405(n), which is defined to be the nth Dowling number corresponding to $m = 2$. For $n \in \mathbb{N}$ define a square matrix A_n of size n as follows:

$$(A_n)_{ij} = \begin{cases} 1, & \text{if } i < j-1, \\ -1, & \text{if } i = j-1, \\ \binom{n-j}{i-j}, & \text{if } i \geq j. \end{cases}$$

The statement in the following theorem was conjecture by Irwin in A007405. The conjecture was also mentioned in [1].

Theorem 1. *Let* $n \in \mathbb{N}_0$. *Then* $\det(A_{n+1}) = a(n)$.

Proof. Set $d_n = \det(A_{n+1})$. Since $A_1 = (1)$, we have $\det(A_1) = 1 = a(0)$, and the assertion holds in this case. Thus, we assume that $n \geq 1$ and therefore analyze A_n for $n \geq 2$. For $j = n, n-1, \ldots, 2$, perform on A_n the column operations $C_j \leftarrow C_j - C_{j-1}$. Denote the resulting matrix by $\widetilde{A}_n$. It is not hard to verify that

$$(\widetilde{A}_n)_{ij} = \begin{cases} \binom{n-1}{i-1}, & \text{if } j = 1, \\ 0, & \text{if } i < j-2, \\ 2, & \text{if } i = j-2, \\ -2, & \text{if } i = j-1, \\ -\binom{n-j}{i-j+1}, & \text{if } i \geq j \geq 2. \end{cases}$$

In particular, the last row of $\widetilde{A}_n$ is $(1, 0, \ldots, 0)$. Therefore, expanding $\det(\widetilde{A}_n)$ along the last row gives

$$\det(A_n) = \det(\widetilde{A}_n) = (-1)^{n+1} \det(B_{n-1}),$$

where B_{n-1} is the square matrix of size $n-1$ obtained by deleting the last row and the first column of $\widetilde{A}_n$. Set $b_0 = 1$ and, for $n \geq 1$, set $b_n = \det(B_n)$.

Thus, $d_n = (-1)^n b_n$. By Lemma 1,

$$b_{n+1} = -b_n - \sum_{k=0}^{n} (-1)^k \binom{n}{k} 2^k b_{n-k}. \tag{1}$$

Multiplying both sides of (1) by $(-1)^{n+1}$ and using $d_n = (-1)^n b_n$, we obtain,

$$d_{n+1} = d_n + \sum_{k=0}^{n} \binom{n}{k} 2^{n-k} d_k. \tag{2}$$

Let $F(x) = \sum_{n=0}^{\infty} d_n \frac{x^n}{n!}$ be the exponential generating function of the sequence $(d_n)_{n \in \mathbb{N}_0}$. Multiplying (2) by $x^n/n!$ and summing over $n \in \mathbb{N}_0$ yields

$$F'(x) = F(x) + F(x) \sum_{m=0}^{\infty} \frac{(2x)^m}{m!} = (1 + e^{2x}) F(x).$$

Since $F(0) = d_0 = 1$,

$$F(x) = \exp\left(\int_0^x (1 + e^{2t}) dt \right) = \exp\left(x + \frac{e^{2x} - 1}{2} \right),$$

exactly the exponential generating function of the Dowling numbers corresponding to $m = 2$ (e.g., [2, Theorem 7]). $\square$

Lemma 1. *For $n \in \mathbb{N}$ let B_n be the square matrix of size n defined by*

$$(B_n)_{ij} = \begin{cases} 0, & \text{if } i < j - 1, \\ 2, & \text{if } i = j - 1, \\ -2, & \text{if } i = j, \\ -\binom{n-j}{i-j}, & \text{if } i > j. \end{cases} \tag{3}$$

Set $b_0 = 1$ and $b_n = \det(B_n)$. Then, for every $n \in \mathbb{N}_0$,

$$b_{n+1} = -b_n - \sum_{k=0}^{n} (-1)^k \binom{n}{k} 2^k b_{n-k}. \tag{4}$$

Proof. Let $n \in \mathbb{N}_0$. We expand $\det(B_{n+1})$ along the first column. For $1 \leq k \leq n+1$ let M_k denote the minor obtained from B_{n+1} by deleting the first column and the kth row. It is not hard to see that $M_1 = B_n$. Furthermore, $(B_{n+1})_{11} = -2$ and, for $2 \leq k \leq n+1$, we have $(B_{n+1})_{k1} = -\binom{n}{k-1}$. Thus,

$$b_{n+1} = -2b_n - \sum_{k=1}^{n} (-1)^k \binom{n}{k} \det(M_{k+1}). \tag{5}$$

Let $1 \leq k \leq n$ and consider M_{k+1}. We claim that M_{k+1} has the following representation as a block matrix:

$$M_{k+1} = \begin{pmatrix} P_k & 0 \\ * & Q_{n-k} \end{pmatrix},$$

where P_k and Q_{n-k} are square matrices of sizes k and $n-k$, respectively. To justify the zero matrix, let $1 \leq i \leq k$ and $k+1 \leq j \leq n$. Then $i \leq k < k+1 \leq j+1-1$. Thus, $(M_{k+1})_{i,j} = (B_{n+1})_{i,j+1} = 0$. It follows that

$$\det(M_{k+1}) = \det(P_k)\det(Q_{n-k}). \tag{6}$$

We now claim that $\det(P_k) = 2^k$. To this end, let $1 \leq i, j \leq k$. We have $(P_k)_{i,j} = (B_{n+1})_{i,j+1}$ and the latter is equal to 0 if $i < j+1-1 = j$ and to 2 if $i = j+1-1 = j$. It follows that P_k is a lower-triangular matrix with all diagonal entries equal to 2. Thus, $\det(P_k) = 2^k$, as asserted.

We now claim that $Q_{n-k} = B_{n-k}$ and therefore $\det(Q_{n-k}) = \det(B_{n-k})$. To this end, let $1 \leq i, j \leq n-k$. We have $(Q_{n-k})_{i,j} = (B_{n+1})_{i+k+1,j+k+1}$. Now, each of the conditions in (3) holds for i and j if and only if it holds for $i+k+1$ and $j+k+1$. Regarding the fourth condition, notice that $-\binom{n+1-(j+k+1)}{i+k+1-(j+k+1)} = -\binom{n-k-j}{i-j}$. Thus, for the entries of Q_{n-k} exactly the same conditions apply as for those of B_{n-k} and hence $Q_{n-k} = B_{n-k}$, as asserted.

From (6) with the two results just obtained, we conclude that $\det(M_{k+1}) = 2^k b_{n-k}$. Substituting this into (5) yields exactly (4). □

Example 1.

$$A_6 = \begin{pmatrix} 1 & -1 & 1 & 1 & 1 & 1 \\ 5 & 1 & -1 & 1 & 1 & 1 \\ 10 & 4 & 1 & -1 & 1 & 1 \\ 10 & 6 & 3 & 1 & -1 & 1 \\ 5 & 4 & 3 & 2 & 1 & -1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}, \quad \widetilde{A}_6 = \begin{pmatrix} 1 & -2 & 2 & 0 & 0 & 0 \\ 5 & -4 & -2 & 2 & 0 & 0 \\ 10 & -6 & -3 & -2 & 2 & 0 \\ 10 & -4 & -3 & -2 & -2 & 2 \\ 5 & -1 & -1 & -1 & -1 & -2 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

$$B_5 = \begin{pmatrix} -2 & 2 & 0 & 0 & 0 \\ -4 & -2 & 2 & 0 & 0 \\ -6 & -3 & -2 & 2 & 0 \\ -4 & -3 & -2 & -2 & 2 \\ -1 & -1 & -1 & -1 & -2 \end{pmatrix}.$$

References

[1] A. Barner, A. Buck, J. Elder, P. E. Harris, and A. Simpson, Flattened Stirling permutations and type B set partitions, *MSU Combinatorics and Graph Theory Seminar*, (2023). Available at https://users.math.msu.edu/group/combinatorics-graph-theory-seminar/harry.pdf.

[2] M. Benoumhani, On Whitney numbers of Dowling lattices, *Discrete Math.* **159** (1996), 13–33.

[3] N. J. A. Sloane, The On-Line Encyclopedia of Integer Sequences, OEIS Foundation Inc., https://oeis.org.