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Q-D Tables

and

Zero - Squares

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Notation

We are interested mainly in sequences of ordinary integers; however, it is convenient to formulate the results for an arbitrary integral domain (commutative ring with unity, no zero divisors). We use the fact that such a domain may be extended into polynomials in one variable, and factored into residues modulo a prime ideal, both of which are again integral domains; and finally embedded in its field of fractions, in which we may perform elementary linear algebra.

Readers inconvenienced by this abstraction may consult say (29) or (31); or else pretend that we are dealing with ordinary integers, polynomials, integers modulo p , rational functions, etc.

There is no need for us to distinguish between the elements of these various domains; they are therefore all denoted by italic capitals. Ordinary integers are denoted by italic small letters; vectors and matrices by bold face.

Linear Recurrences and Q-D Tables

A "linear recurring sequence" or LRS \underline{s} of order r is a sequence of elements s_n , for which a vector \underline{A} and integer $r \geq 0$ exist such that, for all n ,

$$\sum_{i=0}^r A_i s_{n+i} = 0.$$

For a "polynomial sequence" \underline{T} of degree $r-1$ there exists \underline{B} such that

$$T_n = \sum_{j=0}^{r-1} B_j n^j.$$

In elementary difference calculus (30) it is shown that, for all n ,

$$\Delta^r(T_n) = 0;$$

from which it follows that \underline{T} is an LRS with the particular recurrence

$$A_i = (-)^{r-i} \binom{r}{i} \quad (\text{binomial coefficients}).$$

The "difference table" of a sequence has for its m -th row the sequence $\Delta^m(T_n)$, easily constructed from the recursion

$$\Delta^0(T_n) = T_n;$$

$$\Delta^{m+1}(T_n) = \Delta(\Delta^m(T_n)) = \Delta^m(T_{n+1}) - \Delta^m(T_n).$$

Polynomial sequences are characterised by their difference table degenerating to zero for $m \geq r$.

This motivates us to look for an analogous table whose degeneration characterises a general LRS. To this end we define

(1) Definition $S_{mn} = 0$ if $m < 0$,

$$= \begin{vmatrix} S_n & S_{n+1} & \dots & S_{n+m-1} \\ S_{n-1} & S_n & & \\ \vdots & & \ddots & \\ S_{n-m+1} & & S_n & \end{vmatrix}_{m \times m} \quad \text{if } m \geq 0;$$

with a sign change of $(-)^{\binom{m}{2}}$ these are known as Hankel determinants. The array of S_{mn} we call the "Q-D table" of S , in deference to Retishausen, from whose Q-D algorithm (30) we lifted the idea. We often implicitly assume $m \geq 0$.

(2) Lemma Iff $S_{mn} \neq 0$ and $S_{m+1n} = 0$, there is a (finite) LRS, of order precisely m , spanning S_{n-m}, \dots, S_{n+m} . Its coefficients A are unique up to a common factor. Iff further $S_{m+1n} = S_{m+1n+1} = \dots = S_{m+1n+k-1} = 0$ but $S_{m+1n-1}, S_{m+1n+k} \neq 0$, the LRS spans precisely $S_{n-m}, \dots, S_{n+m+k-1}$ and no further.

Proof: set up $m+1$ homogeneous equations for A , with a unique solution iff of rank m , QED.

(3) Corollary \underline{S}_m is an LRS of order r iff row r of its Q-D table does not vanish, but row $r+1$ (and all later rows) does. That is, the Q-D table is the desired analogue.

We next enquire whether it possesses a simple recursive construction. For this we need to be able to evaluate a determinant by pivotal condensation about a minor (28):

(4) Lemma Given an $m \times m$ matrix $\underline{A} = [A_{ij}]$ and an arbitrary $(m-h) \times (m-h)$ minor \underline{C}_m , define D_{ij} to be the [minor] with the rows and columns of \underline{C}_m , together with the i th row and j th column not occurring in \underline{C}_m , and let $\underline{D}_m = [D_{ij}]$; Then

$$|\underline{A}_m| \times |\underline{C}_m|^h = |\underline{D}_m| \times |\underline{C}_m|.$$

Proof: Multiply by $|\underline{C}_m|$ every row i of \underline{A}_m not itself in \underline{C}_m , thus multiplying $|\underline{A}_m|$ by $|\underline{C}_m|^h$; then for these i and all j , replace A_{ij} by the [minor] consisting of \underline{C}_m with row i and column j adjoined, leaving $|\underline{A}_m|$ unchanged since this is a row operation. Each column j in \underline{C}_m now has zeros on the rows not in \underline{C}_m ,

these elements being determinants with two equal columns; so the result may be decomposed into $|C| \times |D|$, QED.

Now set $h=2$, $|A| = S_{m+1,n}$, $|C| = S_{m+1,n}$ in (4); then

$$|D| = \begin{vmatrix} S_{mn} & S_{m+1,n} \\ S_{m+1,n} & S_n \end{vmatrix},$$

so rearranging and cancelling $|C|$,

(5) Theorem $S_{mn}^2 = S_{m+1,n}S_{m-1,n} + S_{m+1}S_{m-1}$.
This immediately provides a recursive construction of row $m+1$ in terms of rows m and $m-1$; except when $S_{m-1,n} = 0$, whereby hangs an intriguing tale.

Zero Squares

(6) Theorem Zero elements $S_{mn} = 0$ of a Q-D table occur in square $g \times g$ clumps with nonzero borders. Furthermore, if S_{mn} is considered as a matrix rather than a determinant, its nullity equals its distance h from the border.

Proof: We have observed (2) that the occurrence of a zero implies that some segment of S is an LRS; let this be of order t , and span precisely S_1, \dots, S_k , by shifting the origin of S . Define g by $k = 2t + g$,

where $g > 0$ since $2r+1$ consecutive elements are required to specify the LRS; and notice that no LRS of smaller order may span any $2r+1$ of s_1, \dots, s_k , otherwise the whole segment would have order $< r$.

Let S_{mn} lie within the square $r \leq (m, n) \leq r+g+1$, and define its "depths" h by

$$h = \min(m-r, n-r, r+g+1-m, r+g+1-n).$$

Consider those rows of (the matrix) S_{mn} which contain only s_1, \dots, s_k ; it is geometrically obvious that there are just $r+h$ such rows. (7), so they span at least $2r+1$ elements of the LRS, whence their dimension is r and nullity h .

Now consider a row i of S_{mn} which contains s_0 in column j , restricting the attention to columns $j, \dots, j+r$. In these columns (and therefore overall) row i is independent of rows $i-r, \dots, i-1$; since only s_1, \dots, s_k occur in the latter and s_0 does not fit the LRS. And rows $i-1, \dots, i-r-1$ do depend on the latter rows, for the same reason (notice that there are $r+h$ columns in the middle of S_{mn} containing only s_1, \dots, s_k). So row i is independent of all previous rows; and the rows containing s_0 ,

and similarly S_{k+1} , have nullity 0.

The nullity of S_{mn} is therefore h . Along the edges of the square $h = 0$; the determinant S_{mn} is therefore nonzero, and similarly it is zero properly inside the square. The interior consists of g^2 elements, and we call it a $g \times g$ "zero-square". QED.

See (19) for an example. Notice that some edges of the zero-square may lie at infinity: for instance when $m < 0$, or $m > r$ and \tilde{S} is an LKS of order r .

$$(7) \quad \begin{array}{l} r=1 \\ g=6 \\ k=8 \\ m=6 \\ n=4 \\ h=2 \end{array} \quad S_{mn} = \left| \begin{array}{cccccc} S_4 & S_5 & S_6 & S_7 & S_8 & S_9 \\ S_3 & S_4 & S_5 & S_6 & S_7 & S_8 \\ S_2 & S_3 & S_4 & S_5 & S_6 & S_7 \\ S_1 & S_2 & S_3 & S_4 & S_5 & S_6 \\ S_0 & S_1 & S_2 & S_3 & S_4 & S_5 \\ S_{-1} & S_0 & S_1 & S_2 & S_3 & S_4 \end{array} \right| \quad \begin{array}{l} \text{indep. rows} \\ \text{3 LRS rows,} \\ \text{dim} = 1 \end{array}$$

$\underbrace{\hspace{10em}}$

2 cols show row 6 indep.,
 $i=6 \ j=2$

(8) Corollary Elements S_{mn} divisible by a prime P clump together in squares with borders coprime to P . Furthermore, if S_{mn} is at depths h from the border, it is divisible by P^h .

Proof: Apply (6) to the domain of residues modulo P : zero-squares in the residue table correspond to P -squares in the original table.

Since the nullity of S_{mn} is h , every (minor) of size $m-h+1$ is zero mod p , whereas some (minor), say $|C_{mn}|$, is nonzero. By (4),

$$S_{mn} = |D|_{h \times h} \times |C_{mn}|^{1-h}$$

where p divides every D_{ij} but not $|C_{mn}|$. So p^h divides S_{mn} , QED.

Border Theorems

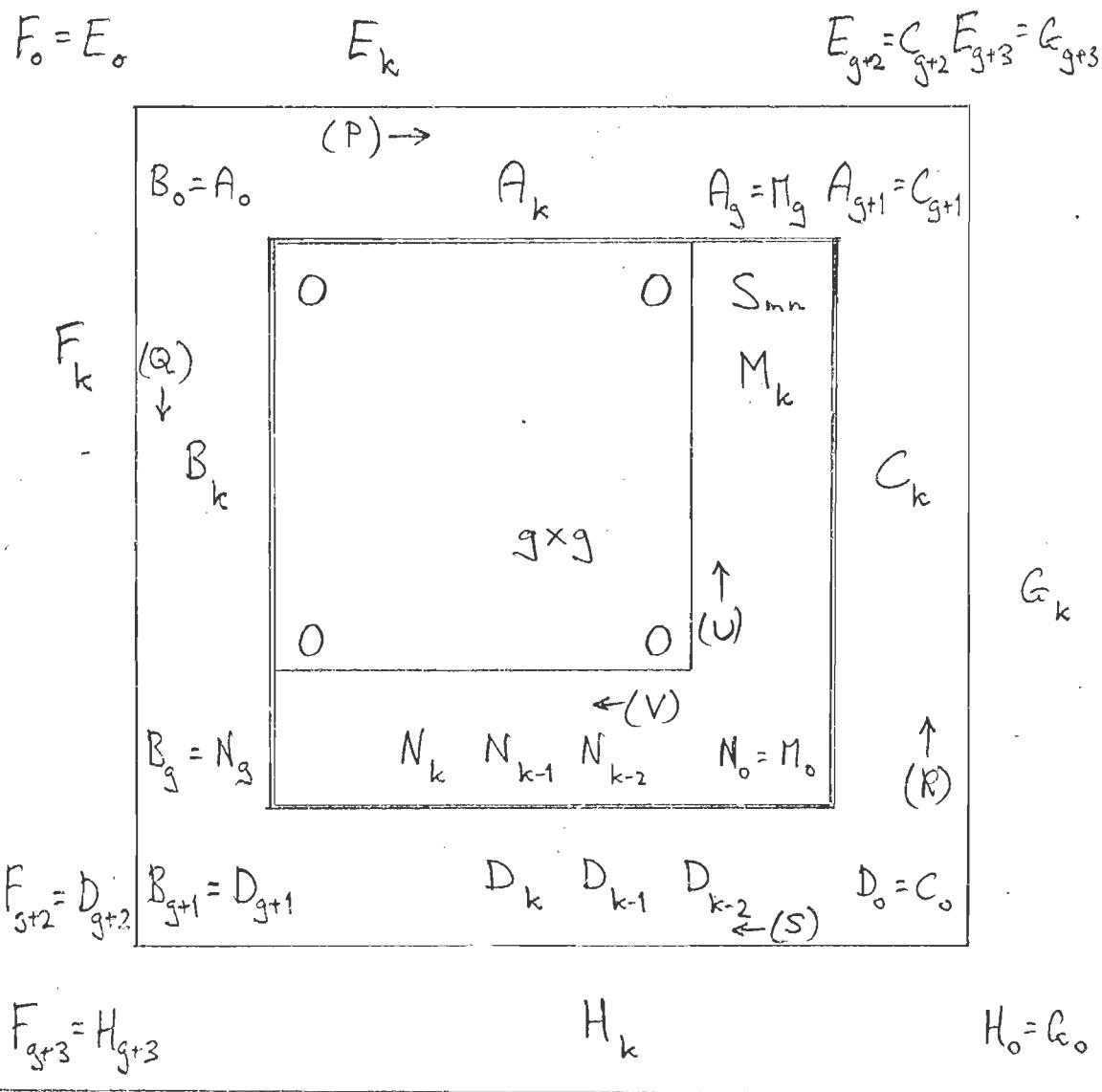
To complete our program we require recursions relating the bottom edge of a border to its other three edges: one (18) for the "inner" border next to the zeros, one (16) for the "outer" border immediately outside it. Having circumvented a zero-square with these we can revert to (5).

The proof borrows from algebraic geometry. If S gives a $g \times g$ zero-square whose top-right corner is S_{mn} , we perturb S to a new sequence S' :

$$\begin{aligned} S'_l &= S_l & \text{for } l \neq n+m-1; \\ &= S_{n+m-1} - (-)^m X & \text{for } l = n+m-1; \end{aligned}$$

where X is transcendental. The ' will usually be omitted. The elements are now polynomials in X , with a $(g-1) \times (g-1)$ zero-square since S_{mn} etc are now nonzero. We inductively prove an approximation to the theorem, then "let $X \rightarrow 0$ " — that is, specialise to the constant coefficient — for the exact theorem.

(9)



The situation is diagrammed in (9). The original $g \times g$ square had inner border $A = A_1, B = A_2, C = C_1, D = C_2$ and outer border $E = E_1, F = E_2, G = G_1, H = G_2$; the polynomial square has inner border $\underline{A}, \underline{B}, \underline{M} = \underline{M}_1, \underline{N} = \underline{M}_2$. We shall use subscript $i=1,2$ to select an edge, and k to select an element along it, from an origin in the top left or bottom right corner: e.g. if $i=1, k=g-1$, $\underline{M}_{ik} = \underline{M}_{g-1} = S_{mn}$.

By (5) the inner boundary edges satisfy

$$A_{ik}^2 = A_{i,k-1} A_{i,k+1}, \quad M_{ik}^2 = M_{i,k-1} M_{i,k+1}, \quad i=1,2;$$

they are therefore geometric sequences whose ratios we denote by

$$P = P_1, \quad U = U_1, \quad Q = P_2, \quad V = U_2 \quad \text{resp.}$$

We also abbreviate $-P$ to \bar{P} , etc. Then

$$(10) \quad \underline{\text{Lemma}} \quad A_{ik} = A_{i,0} P_i^k.$$

$$(11) \quad \underline{\text{Lemma}} \quad U_i = e P_i / X, \quad \text{where } e = (-)^{(i-1)(g-1)}.$$

$$\text{Proof: } M_{g-1} = S'_{mn} = \begin{vmatrix} S_n & \dots & S_{n+m-1} & (-)^m X \\ \vdots & \ddots & \vdots & \vdots \\ S_{n-m+1} & \dots & S_n & \end{vmatrix}.$$

$$= \begin{vmatrix} S_{n-1} & \dots & S_{n+m-2} \\ \vdots & \ddots & \vdots \\ S_{n-m+1} & \dots & S_{n-1} \end{vmatrix} X + \begin{vmatrix} S_n & \dots & S_{n+m-1} \\ \vdots & \ddots & \vdots \\ S_{n-m+1} & \dots & S_n \end{vmatrix}$$

by (1) and expanding the $m \times m$ determinant;

$$= S_{m-1,n-1} X + S_{mn} = A_{g-1} X$$

since S_{mn} was originally zero. So

$$M_g / M_{g-1} = A_g / A_{g-1} X \quad \text{since } M_g = A_g,$$

$$U = P/X.$$

Similarly $V = (-)^{g-1} Q / X$, the determinant being of size $m+g-1$. Both results are contained in (11), QED.

$$(12) \quad \underline{\text{Corollary}} \quad M_{ik} = M_{ig} U_i^{k-g} = A_{ig} (X/e P_i)^{g-k}.$$

C and D are not geometric, but we can still define the ratio

$$R_i = C_{i,1} / C_{i,0}$$

and calculate how closely they approximate to the geometric:

(13) Lemma For $0 \leq k \leq g+2$, $C_{ik} = C_{i0} R_i^{-k}$

$$= (R_i/P_i)^{g+3-k} e^{k-g^2} G_{ik} X^{g-k+2} + O(X^{g-k+3}),$$

where " $O(X^d)$ " means X^d times some rational function which is finite when $X \rightarrow 0$.

Proof by induction on k : Trivial for $k=0$, by definition for $k=1$. For $k \geq 2$,

$$\begin{aligned} C_{ik} &= C_{i,k-2}^{-1} (C_{i,k-1}^{-2} - M_{i,k-2} G_{ik}) \quad \text{by (5)}; \\ &= C_{i0}^{-1} R_i^{2-k} C_{i0}^{-2} R_i^{2k-2} - C_{i0}^{-1} R_i^{2-k} A_{ig} (X/eP_i)^{g-k+2} \\ &\quad + O(X^{g-k+3}), \quad \text{by (12) and inductive ass.}; \end{aligned}$$

and $C_{i0}^{-1} A_{ig} = A_{i0} P_i^g / C_{i0} = P_i^g R_i^{g+1} P_i^{-g-1}$, looking at (9). QED.

(14) Theorem $PS/QR = (-)^g + O(X^{g+1})$.

Proof: $PV/QU = (-)^{g-1}$ by (11);

$VR/US = N_1 C_1 / M_1 D_1$ by defn. of ratios;

$$= (-M_1 D_1 + M_0^2) / M_1 D_1 \quad \text{by (5)};$$

$$= -1 + O(X^{2g-g+1}) \quad \text{by (12)};$$

collecting,

$$PS/QR = (PV/QU)(VR/US)$$

$$= (-)^g + O(X^{g+1}), \quad \text{QED.}$$

(15) Theorem For $0 \leq k \leq g+3$,

$$P^{g+3} (P^{-k} E_k - \bar{Q}^{-k} F_k) = \\ \bar{R}^{g+3} (\bar{R}^{-k} G_k - S^{-k} H_k) + O(x).$$

Proof : case $k = g+3$: this reduces to

$$E_{g+3} - (P/\bar{Q})^{g+3} F_{g+3} = G_{g+3} - (\bar{R}/S)^{g+3} H_{g+3} + O(x),$$

which is evident from (9) and (14).

Case $3 \leq k \leq g+2$: by induction on g . For $g > 0$, by ind. ass. or the previous case (whence $k \geq 3$),

$$P^{g+3} (P^{-k} E_k - \bar{Q}^{-k} F_k) = \\ P \bar{U}^{g+2} [\bar{U}^{-k} C_k - V^{-k} D_k] + O(x) \\ = P(\bar{P}/x)^{g+2} [(\bar{P}/x)^{-k} C_0 R^k - (e\bar{Q}/x)^{-k} D_0 S^k] \\ - P(\bar{P}/x)^{g+2} [(\bar{P}/x)^{-k} (R/P)^{g+3-k} G_k X^{g-k+2} - \\ (e\bar{Q}/x)^{-k} (S/Q)^{g+3-k} e^{k-g-2} H_k X^{g-k+2}] + O(x),$$

where $e = (-)^{g-1}$, by (11), (13) (whence $k \leq g+2$), rearranging;

$$= (-)^{g+2-k} P^{g+3} C_0 X^{k-g-2} [(R/P)^k - (S/Q)^k (-)^{gk}] \\ + \bar{P}^{g+3} (R/P)^{g+3} [\bar{R}^{-k} G_k - (PS/RQ)^{g+3} S^{-k} H_k] + O(x),$$

using $e^{g+2} = 1$, $C_0 = D_0$, and extracting powers of X ; $(-)^k$ extracted from first [...];

$$= O(x^{k-1}) + \bar{R}^{g+3} [\bar{R}^{-k} G_k + S^{-k} H_k] + O(x),$$

by (11); then use $k > 1$.

Case $0 \leq k \leq 2$: reflecting left into right in (9) and applying the previous two cases, we get by symmetry a new theorem, which may be written in terms of the unreflected notation by the transformations

$$\begin{array}{lll}
 k \rightarrow g+3-k & F_k \rightarrow G_k & P \rightarrow P^{-1} \\
 & G_k \rightarrow F_k & Q \rightarrow R^{-1} \\
 & & R \rightarrow Q^{-1} \\
 & & S \rightarrow S^{-1}
 \end{array}$$

The result is that for $0 \leq k \leq g$,

$$P^{-k} E_k - \bar{Q}^{-k} F_k =$$

$$(\bar{R}/P)^{g+3} \bar{R}^{-k} G_k - (S/\bar{Q})^{g+3} S^{-k} H_k + O(x),$$

which reduces to (15) again by (11) and multiplying by P^{g+3} . This disposes of all remaining cases except $g=0, 1; k=2$. However, for $k=2$ the induction on g above may be reversed; so those follow from $g=k=2$, QED.

Finally, let $X \rightarrow 0$ in (15) and (14); then in the original table,

(16) Theorem (outer border)

$$P^{g+3}(P^{-k} E_k - \bar{Q}^{-k} F_k) = \bar{R}^{g+3}(\bar{R}^{-k} G_k - S^{-k} H_k);$$

(17) Theorem (inner border) $PS/QR = (-)^g$;

(18) Corollary $A_k D_k / B_k C_k = (-)^{gk}$ by (10).

Example

(19)

1	1	1	1	1	1	1	0
0	1	2	3	4	5	0	1
-5	1	1	1	1	25	-5	2
6.9	6	0	0	-6	6.21	6.9	3
-6^2.12	6^2	0	0	6^2	6^2.18	-6^2.12	4
6^3.14	6^3	-6^3	6^3	-6^3	6^3.16	6^3.14	5
-6^4.15	6^4.15	-6^4.15	6^4.15	-6^4.15	6^4.15	-6^4.15	6
0	0	0	0	0	0	0	7

A numerical example of a Q-D table (19) illustrates the results. The original sequence S is the natural numbers modulo 6, so that the whole table is horizontally periodic. Notice the 2×2 zero-square (6) at S_{32} ; the infinite zero-square at row 7, S being an LRS of order 6 (3); the 2×2 5-square at S_{15} (7); the infinite 2- and 3-squares at row 3.

Rows 2 to 4 and half of rows 5 and 6 may be computed from (5), e.g.

$$S_{15} = 5, S_{24} = 1, S_{25} = 25, S_{26} = -5,$$

$$S_{35} = 6.21 = S_{15}^{-1}(S_{25}^2 - S_{24}S_{26}).$$

The inaccessible part of row 5 falls to (18), e.g.
 $k=g=2, D=S_{53},$

$$A = 1, B = 6, C = 6^2, D = 6^3 = (-)^4 BC/A;$$

and of row 6 to (16), e.g. $H = S_{63},$

$$P = 1, \bar{Q} = 6, \bar{R} = -6^{-1}, S = -1;$$

$$E = 2, F = 6.9, G = 6^2.18, H = 6^4.15$$

$$= S^2 \bar{R}^{-2} G - S^2 \bar{R}^{-5} P^5 (P^2 E - \bar{Q}^{-2} F).$$

A Short-Cut to a Fallacy

A less devious way to prove (15) seems to be to replace some less distant element, such as the original bottom-right zero M_0 in (9), by Z , which is implicitly a polynomial in some other transcendental X assumed to have been introduced (as above) into some appropriate element of the original sequence S . For example,

(20) Lemma In the notation of the attached figure,

$$\begin{array}{l} ED^2 + HA^2 - Z(EH + AD) = \\ FC^2 + GB^2 - Z(FG + BC). \end{array} \quad \begin{array}{c} E \\ L \\ F \\ N \\ A \\ B \\ Z \\ D \\ K \\ C \\ M \\ H \end{array}$$

Proof: Expressing $LKHN$ in two ways by (5),

$$(A^2 - EZ)(D^2 - HZ) = (B^2 - FZ)(C^2 - GZ);$$

whence $O =$

$$(B^2C^2 - A^2D^2) + Z(ED^2 + HA^2 - FC^2 - GB^2) - Z^2(EH - FG).$$

Substituting for

$$(21) \quad BC + AD = Z^2 \quad \text{by (5)}$$

and cancelling Z gives (20), QED.

(20) with $Z \rightarrow 0$ is (16) with $g=1, k=2$. (21) and (20) together suggest a system of cruciform identities of increasing span, contradicted by closer examination of (16).

$$\begin{array}{ccccccccc}
 & 1 & 1 & 1 & 1 & 1 & & 1 & 1 & 1 & 1 & 1 \\
 S_{-3} & S_{-2} & 0 & S_0 & 0 & S_2 & S_3 & S_{-3} & S_{-2} & 0 & 0 & 0 & S_2 & S_3 \\
 * & * & Z & * & C & & & * & 0 & 0 & 0 & C & & \\
 * & 0 & * & 0 & * & G & & * & 0 & 0 & 0 & * & G & \\
 * & * & D & * & * & & & * & * & D & * & * & & \\
 & & H & & & & & & & & & & H & \\
 \end{array}$$

(22)

(23)

Now consider figs. (22), (23), where asterisks denote arbitrary nonzero elements. In the first, by (20)

$$Z(CH - DG) = 0 ;$$

whence cancelling Z as before,

$$(24) \quad CH = DG .$$

Letting $Z \rightarrow 0$, by (6) the first becomes the second; but (24) is now false!

The trouble is that, since $S_{-13} = S_{13} = 0$ in (22), there are two (distinct) second-order LRS's spanning S_{-3}, \dots, S_1 and S_{-1}, \dots, S_3 by (2). So not only is $S_0 = X$ (say) a function of X but the whole of row 1, and hence the whole diagram for $m > 0$, are too. By the recurrences, when $X \rightarrow 0$ so does row 1 and all the rest, and the "limit" of fig (22) is not fig (23) but a great big zero-square of size at least 7, in which (26) is trivial. To sum up, theorems about (22) fail to converge to theorems about (23) because the manifold of sequences determined by the latter is not on the boundary of the former.

Symmetric Definition

The initial definition (1) of Q-D tables was asymmetric. However (5), (6), (16), (17), which have the symmetry of the square lattice (by third case of (15) proof) could be used instead to generate a "symmetric" Q-D table from any pair of initial sequences, provided they were consistent with these theorems, and sufficient borders were provided for any zero-squares.

(8) shows that pairs of integer sequences do not necessarily have integer tables; so we pose the

(25) Problem Establish conditions under which the Q-D table of a pair of sequences is integer.

The connection between old and new definitions is that

(26) Theorem The Q-D table of \underline{S} is the symmetric Q-D table on $\underline{0}, \underline{1}, \underline{S}$.

Proof: the nontrivial part is to show that we need to specify $S_{-1n} = 0$, which strictly speaking we should have done when proving (15).

By (5)

$$S_{-1n} = S_{1n}^{-1} (1^2 - 1 \cdot 1),$$

which shows $S_{-1n} = 0$ if $S_{1n} \neq 0$. If $S_{1n} = 0$, as in the proof of the border theorems we perturb it to $S'_{1n} = X$, deduce $S_{-1n} = 0$ as above, then let $X \rightarrow 0$. QED.

Finally, as an amusing application of this approach, consider the three infinite rows exemplified for $g=2$ by (27); one could replace every * by 1, for instance. By the border theorems these rows generate an infinite tesselation of $g \times g$ zero-squares in both dimensions; so by (26) there is no sequence S which has these 3 rows anywhere in its (asymmetric) Q-D table, the latter having an infinite zero-square at $m < 0$.

$$\begin{array}{cccccccccc} 0 & 0 & * & 0 & 0 & * & 0 & 0 & * & 0 \\ \dots & * & * & * & * & * & * & * & * & * & \dots \\ 0 & 0 & * & 0 & 0 & * & 0 & 0 & * & 0 \end{array}$$

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