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Next let Q' be the paramount matrix

$$Q' = \begin{bmatrix} 7 & 1 & 2 & 3 \\ 1 & 12 & 4 & 5 \\ 2 & 9 & 15 & 6 \\ 3 & 5 & 6 & 18 \end{bmatrix}.$$

Cederbaum [1] has shown that Q' cannot be displayed as in the hypothesis of Theorem 2. However $Q_{i,r,c} \geq 0$ for $1 \leq i, r, c \leq 4$. Consequently, paramountcy and the condition that $Q_{i,r,c} \geq 0$ are not sufficient to guarantee the unimodular decomposition of Q'.

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Tables of Essential Complementary Partitions

Abstract—Tables of essential complementary partitions of a set S_k are presented for $k = 2, 3, 4, 5$ where k is the number of elements of S_k. They are sufficient for most practical applications.

In a recent paper Chen [1] has shown how the trees of a very complicated graph can be generated by decomposition without redundancies due to duplications. However, his generation formulas require that one first tabulate all the essential complementary partitions of a set S_k for some reasonably large values of k. In this correspondence we shall present these tables for $k=2, 3, 4, 5$.

Let $S_k = \{1, 2, \cdots, k\}$. Let $C[P'(S_k); P''(S_k)]$ be a pair of essential complementary partitions of S_k. Since the pair of complementary partitions $[P'(S_k); P''(S_k)]$ are essential if, and only if, the pair $[P''(S_k); P'(S_k)]$ are essential, it is clear that we do not need to list them all; half would be sufficient. The other half can easily be obtained by interchanging the roles of $P'(S_k)$ and $P''(S_k)$.

For a given set S let $\alpha(S)$ be the number of its elements. Thus $\alpha(S_k) = k$. For a fixed k the set $H(S_k)$ of all possible $C[P'(S_k); P''(S_k)]$ can be generated as follows. We first generate the set $H(P(S_k))$ of all possible partitions $P(S_k)$ of S_k, which in turn can be generated iteratively from $H(P(S_{k-1}))$, $k \geqq 2$, by inserting the integer k to the elements of $P(S_{k-1})$.[1] In $H(P(S_k))$ we form all possible pairs $[P'(S_k); P''(S_k)]$ with the property that $\alpha[P'(S_k) \cup P''(S_k)] = k+1$. Then we use the algorithm proposed in [1] to test the essentiality of $[P'(S_k); P''(S_k)]$. Based on this principle, a computer program was written in Fortran IV for the IBM System 360 Model 44 digital computer [2]. The results are presented in Tables I–III for $k=3$, 4, and 5, respectively. For $k=2$ there are only two $C[P'(S_2); P''(S_2)]$, one of which is $C[\{1, 2\}; \{12\}]$. Thus we have $\alpha(H(S_2)) = 2$, $\alpha(H(S_3)) = 8$, $\alpha(H(S_4)) = 50$, and $\alpha(H(S_5)) = 432$.

To show that these are indeed the right numbers for $H(S_k)$, we use the following argument. Let G' and G'' be the k-node complete graph,

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[1] It also includes the case where $\{k\}$ is added to $P(S_{k-1})$.

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TABLE I

$(P'(S_3); P''(S_3))$ or $(P''(S_3); P'(S_3))$			
(123; 1,2,3)	(1,23; 13,2)	(1,23; 12,3)	(13,2; 12,3)

TABLE II

$(P'(S_4); P''(S_4))$ or $(P''(S_4); P'(S_4))$		
(1234; 1,2,3,4)	(123,4; 14,2,3)	(123,4; 1,24,3)
(123,4; 1,2,34)	(1,234; 13,2,4)	(1,234; 12,3,4)
(1,234; 14,2,3)	(14,23; 13,2,4)	(14,23; 12,3,4)
(14,23; 1,24,3)	(14,23; 1,2,34)	(134,2; 1,23,4)
(134,2; 12,3,4)	(134,2; 1,24,3)	(13,24; 1,23,4)
(13,24; 12,3,4)	(13,24; 14,2,3)	(13,24; 1,2,34)
(124,3; 1,23,4)	(124,3; 13,2,4)	(124,3; 1,2,34)
(12,34; 1,23,4)	(12,34; 13,2,4)	(12,34; 14,2,3)
(12,34; 1,24,3)		

and let G be the graph obtained by superimposing G'' on G'. For each element $C[P'(S_k), P''(S_k)]$ of $H(S_k)$, let p_x' $(x=1, 2, \cdots, u)$ and p_y'' $(y=1, 2, \cdots, v)$ be the elements of $P'(S_k)$ and $P''(S_k)$, respectively. Using the symbols defined in [1], it is not difficult to see that there are

$$\beta' = \prod_{x=1}^{u} [\alpha(p_x')]^{\alpha(p_x')-2} \quad \text{and} \quad \beta'' = \prod_{y=1}^{v} [\alpha(p_y'')]^{\alpha(p_y'')-2}$$

u-trees $t(P'(S_k))$ in G' and v-trees $t(P''(S_k))$ in G'', respectively. Thus for each $C[P'(S_k); P''(S_k)]$ there are $\beta = \beta'\beta''$ corresponding trees in G. Summing over all β for all the elements of $H(S_k)$, we obtain the number of trees of G, which is known to be $2(2k)^{k-2}$, $k \geqq 2$ (see, for example, [5]).

From Table II we find that there are 12 partitions, each containing an element having three integers, and 12 partitions, none of their elements having more than two integers. Thus we have

$$2(4^2 + 12 \cdot 3^1 + 12 \cdot 1) = 2 \cdot 64 = 2(2 \cdot 4)^{4-2}.$$

Similarly, from Table III we obtain

$$2(5^3 + 20 \cdot 4^2 + 120 \cdot 3^1 + 60 \cdot 1 + 15 \cdot 3^1 \cdot 3^1) = 2000 = 2(2 \cdot 5)^{5-2}.$$

For $k=6$ there are 4802 partitions in $H(S_6)$. In general, for $k \geqq 2$ there are

$$2(k+1)^{k-2}$$

essential complementary partitions of S_k. Since a k-node graph has at most k^{k-2} trees, it is clear that it is not advantageous to decompose a graph along all of its nodes in the form of parallel connections, as suggested by Arango *et al.* [4].

The elements of $H(S_6)$ have also been generated. However, they are too numerous to be listed here. For the interested reader, the computer program together with the table for $H(S_6)$ may be found in [2].

Finally, we mention that for most practical networks such as a cascade of multistage amplifiers, the tables presented in this correspondence are sufficient for generating trees by decomposition in their corresponding graphs, since usually there are only a few feedback loops.

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III. Singular Paramount Matrices

The following theorem provides a connection between paramount and totally unimodular matrices.

Theorem 1

Let Q be a $p\times p$ paramount matrix of rank s satisfying

$$Q\begin{pmatrix}1 & \cdots & s\\ 1 & \cdots & s\end{pmatrix} \neq 0.$$

Then Q can be expressed as

$$Q = A^t Q_s A$$

where A is a $p\times s$ totally unimodular matrix and Q_s is the submatrix formed from the first s rows and columns of Q.

Proof: Partition Q as

$$Q = \left[\begin{array}{c|c} Q_s & Q_{12} \\ \hline Q_{21}^t & Q_{22}\end{array}\right]. \tag{10}$$

Set

$$T = \left[\begin{array}{c|c} Q_s^{-1} & 0_{s\times(p-s)} \\ \hline Q_{12}^t & 1_{p-s}\end{array}\right]$$

and form TQ:[1]

$$TQ = \left[\begin{array}{c|c} 1_s & Q_s^{-1}Q_{12} \\ \hline 0_{(p-s)\times s} & Q_{22} - Q_{12}^t Q_s^{-1}Q_{12}\end{array}\right].$$

Since $\det(T)\neq 0$, the rank of TQ is s and therefore

$$Q_{22} - Q_{12}^t Q_s^{-1} Q_{12} = 0_{(p-s)\times(p-s)}. \tag{11}$$

Setting $A=[1_s|\,Q_s^{-1}Q_{12}]$ and using (1) and (2), we can express Q as

$$Q = A^t Q_s A. \tag{12}$$

It remains to show that A is a totally unimodular matrix. Expanding (3) using the Binet–Cauchy formula, we get

$$Q\begin{pmatrix}i_1 & \cdots & i_s\\ j_1 & \cdots & j_s\end{pmatrix} = \det(Q_s) A\begin{pmatrix}1 & \cdots & s\\ i_1 & \cdots & i_s\end{pmatrix} A\begin{pmatrix}1 & \cdots & s\\ j_1 & \cdots & j_s\end{pmatrix}. \tag{13}$$

Using the fact that Q is a paramount matrix in conjunction with (4), it is easy to show that the absolute values of the nonzero $s\times s$ minors of A are all equal; that is, if

$$A\begin{pmatrix}1 & \cdots & s\\ i_1 & \cdots & i_1\end{pmatrix} \neq 0 \quad\text{and}\quad A\begin{pmatrix}1 & \cdots & s\\ j_1 & \cdots & j_s\end{pmatrix} \neq 0$$

then

$$A\begin{pmatrix}1 & \cdots & s\\ i_1 & \cdots & i_s\end{pmatrix} = A\begin{pmatrix}1 & \cdots & s\\ j_1 & \cdots & j_s\end{pmatrix}.$$

Moreover, the modulus is unity since A contains an $s\times s$ unit matrix. To complete the theorem, we must show that the nonzero minors of A of order less than s have a modulus of unity. That this is true can be seen by making a correspondence between minors whose order is less than s and certain minors of A of order s. Let

$$A\begin{pmatrix}i_1 & \cdots & i_r\\ j_1 & \cdots & j_r\end{pmatrix}$$

be such that $r<s$. Let $i_1', \cdots, i_{s-r}'$ be such that $i_1', \cdots, i_{s-r}'$ and $i_1, \cdots, i_r$ form a complete set of indices on the integers $1, \cdots, s$. The sth-order minor corresponding to

$$A\begin{pmatrix}i_1 & \cdots & i_r\\ j_1 & \cdots & j_r\end{pmatrix}$$

is

$$A\begin{pmatrix}i_1' & \cdots & i_{s-r}' i_1 & \cdots & i_r\\ i_1' & \cdots & i_{s-r}' j_1 & \cdots & j_r\end{pmatrix}.$$

It follows from the structure of A that

$$\left|A\begin{pmatrix}i_1 & \cdots & i_r\\ j_1 & \cdots & j_r\end{pmatrix}\right| = \left|A\begin{pmatrix}i_1' & \cdots & i_{s-r}' i_1 & \cdots & i_r\\ i_1' & \cdots & i_{s-r}' j_1 & \cdots & j_r\end{pmatrix}\right|.$$

IV. Unimodular Decomposition

Theorem 2

Let $Q=[q_{ij}]=ADA^t$, where $A=[a_{ij}]$ is a $p\times b$ totally unimodular matrix and $D=[d_{ij}]$ is a $b\times b$ diagonal matrix with positive diagonal terms. Then

$$Q_{i.r.c} = q_{ii} + |q_{rc}| - |q_{ri}| - |q_{ic}| \geq 0, \qquad \text{for all } 1 \leq i, r, c \leq p.$$

Proof: The i, jth element of Q is

$$q_{ij} = \sum_{k=1}^{b} d_{kk} a_{ik} a_{jk}.$$

It is well known [1] that for fixed i and j all the nonzero products $a_{ik}a_{jk}$, for $k=1, \cdots, b$, have the same sign. Consequently,

$$|q_{ij}| = \sum_{k=1}^{b} d_{kk} |a_{ik}a_{jk}|.$$

Therefore we can write $Q_{i.r.c}$ as

$$Q_{i.r.c} = \sum_{k=1}^{b} d_{kk}[a_{ik}^2 + |a_{rk}a_{ck}| - |a_{rk}a_{ik}| - |a_{ck}a_{ik}|].$$

We prove $Q_{i.r.c}\geq 0$ by showing that each term in the summation is nonnegative.

Case 1: $a_{ik}=0$; therefore the only contribution is from $|a_{rk}a_{ck}|$, which is nonnegative.

Case 2: $a_{ik}\neq 0$ and $a_{rk}a_{ck}=0$; therefore the term

$$d_{kk}[a_{ik}^2 - |a_{rk}a_{ik}| - |a_{ck}a_{ik}|]$$

is nonnegative since at least one of the terms $|a_{rk}a_{ik}|$ or $|a_{ck}a_{ik}|$ is zero.

Case 3: $a_{ik}\neq 0$ and $a_{rk}a_{ck}\neq 0$; therefore

$$d_{kk}[a_{ik}^2 + |a_{rk}a_{ck}| - |a_{rk}a_{ik}| - |a_{ck}a_{ik}|] = 0.$$

We conclude, therefore, that $Q_{i.r.c}\geq 0$ for $1\leq i, r, c\leq p$.

Example

The matrix

$$Q = \begin{bmatrix} 20 & -10 & 0\\ -10 & 35 & -30\\ 0 & -30 & 36\end{bmatrix}$$

is a *paramount* matrix [1]; however Q does not satisfy Theorem 2 since

$$Q_{2.1.3} = 35 + 0 - 10 - 30 = -5.$$

Accordingly, the paramountcy condition and the condition of Theorem 2 are independent.

[1] $0_{i\times j}$ is an $i\times j$ matrix with all zero entries, and 1_i is the $i\times i$ unit matrix.

TABLE III

$(P'(S_5);\ P''(S_5))$ or $(P''(S_5);\ P'(S_5))$		
(12345; 1,2,3,4,5)	(1234,5; 15,2,3,4)	(1234,5; 1,25,3,4)
(1234,5; 1,2,35,4)	(1234,5; 1,2,3,45)	(1235,4; 1,24,3,5)
(1235,4; 14,2,3,5)	(1235,4; 1,2,34,5)	(1235,4; 1,2,3,45)
(123,45; 1,24,3,5)	(123,45; 14,2,3,5)	(123,45; 1,2,34,5)
(123,45; 15,2,3,4)	(123,45; 1,25,3,4)	(123,45; 1,2,35,4)
(1,2345; 12,3,4,5)	(1,2345; 14,2,3,5)	(1,2345; 13,2,4,5)
(1,2345; 15,2,3,4)	(15,234; 12,3,4,5)	(15,234; 14,2,3,5)
(15,234; 13,2,4,5)	(15,234; 1,25,3,4)	(15,234; 1,2,35,4)
(15,234; 1,2,3,45)	(14,235; 12,3,4,5)	(14,235; 1,24,3,5)
(14,235; 13,2,4,5)	(14,235; 1,2,34,5)	(14,235; 15,2,3,4)
(14,235; 1,2,3,45)	(145,23; 12,3,4,5)	(145,23; 1,24,3,5)
(145,23; 13,2,4,5)	(145,23; 1,2,34,5)	(145,23; 1,25,3,4)
(145,23; 1,2,35,4)	(1345,2; 12,3,4,5)	(1345,2; 1,24,3,5)
(1345,2; 1,23,4,5)	(1345,2; 1,25,3,4)	(134,25; 12,3,4,5)
(134,25; 1,24,3,5)	(134,25; 1,23,4,5)	(134,25; 15,2,3,4)
(134,25; 1,2,35,4)	(134,25; 1,2,3,45)	(135,24; 12,3,4,5)
(135,24; 14,2,3,5)	(135,24; 1,23,4,5)	(135,24; 1,2,34,5)
(135,24; 1,25,3,4)	(135,24; 1,2,3,45)	(13,245; 12,3,4,5)
(13,245; 14,2,3,5)	(13,245; 1,23,4,5)	(13,245; 1,2,34,5)
(13,245; 15,2,3,4)	(13,245; 1,2,35,4)	(1245,3; 13,2,4,5)
(1245,3; 1,23,4,5)	(1245,3; 1,2,34,5)	(1245,3; 1,2,35,4)
(124,35; 13,2,4,5)	(124,35; 1,23,4,5)	(124,35; 1,2,34,5)
(124,35; 15,2,3,4)	(124,35; 1,25,3,4)	(124,35; 1,2,3,45)
(125,34; 1,24,3,5)	(125,34; 14,2,3,5)	(125,34; 13,2,4,5)
(125,34; 1,23,4,5)	(125,34; 1,2,35,4)	(125,34; 1,2,3,45)
(12,345; 1,24,3,5)	(12,345; 14,2,3,5)	(12,345; 13,2,4,5)
(12,345; 1,23,4,5)	(12,345; 15,2,3,4)	(12,345; 1,25,3,4)
(13,24,5; 1,235,4)	(13,24,5; 15,23,4)	(13,24,5; 1,23,45)
(13,24,5; 125,3,4)	(13,24,5; 12,35,4)	(13,24,5; 12,3,45)
(13,24,5; 145,2,3)	(13,24,5; 14,25,3)	(13,24,5; 14,2,35)
(13,24,5; 15,2,34)	(13,24,5; 1,25,34)	(13,24,5; 1,2,345)
(1,234,5; 135,2,4)	(1,234,5; 13,25,4)	(1,234,5; 13,2,45)
(1,234,5; 125,3,4)	(1,234,5; 12,35,4)	(1,234,5; 12,3,45)
(1,234,5; 145,2,3)	(1,234,5; 14,25,3)	(1,234,5; 14,2,35)
(134,2,5; 1,235,4)	(134,2,5; 15,23,4)	(134,2,5; 1,23,45)
(134,2,5; 125,3,4)	(134,2,5; 12,35,4)	(134,2,5; 12,3,45)
(134,2,5; 15,24,3)	(134,2,5; 1,245,3)	(134,2,5; 1,24,35)
(124,3,5; 1,235,4)	(124,3,5; 15,23,4)	(124,3,5; 1,23,45)
(124,3,5; 135,2,4)	(124,3,5; 13,25,4)	(124,3,5; 13,2,45)
(124,3,5; 15,2,34)	(124,3,5; 1,25,34)	(124,3,5; 1,2,345)
(14,23,5; 135,2,4)	(14,23,5; 13,25,4)	(14,23,5; 13,2,45)
(14,23,5; 125,3,4)	(14,23,5; 12,35,4)	(14,23,5; 12,3,45)
(14,23,5; 15,24,3)	(14,23,5; 1,245,3)	(14,23,5; 1,24,35)
(14,23,5; 15,2,34)	(14,23,5; 1,25,34)	(14,23,5; 1,2,345)
(123,4,5; 145,2,3)	(123,4,5; 14,25,3)	(123,4,5; 14,2,35)
(123,4,5; 15,24,3)	(123,4,5; 1,245,3)	(123,4,5; 1,24,35)
(123,4,5; 15,2,34)	(123,4,5; 1,25,34)	(123,4,5; 1,2,345)
(12,34,5; 1,235,4)	(12,34,5; 15,23,4)	(12,34,5; 1,23,45)
(12,34,5; 135,2,4)	(12,34,5; 13,25,4)	(12,34,5; 13,2,45)
(12,34,5; 145,2,3)	(12,34,5; 14,25,3)	(12,34,5; 14,2,35)
(12,34,5; 15,24,3)	(12,34,5; 1,245,3)	(12,34,5; 1,24,35)
(1,235,4; 13,2,45)	(1,235,4; 12,3,45)	(1,235,4; 145,2,3)
(1,235,4; 15,24,3)	(1,235,4; 15,2,34)	(15,23,4; 13,2,45)
(15,23,4; 12,3,45)	(15,23,4; 14,25,3)	(15,23,4; 14,2,35)
(15,23,4; 1,245,3)	(15,23,4; 1,24,35)	(15,23,4; 1,25,34)
(15,23,4; 1,2,345)	(1,23,45; 135,2,4)	(1,23,45; 13,25,4)
(1,23,45; 125,3,4)	(1,23,45; 12,35,4)	(1,23,45; 14,25,3)
(1,23,45; 14,2,35)	(1,23,45; 15,24,3)	(1,23,45; 15,2,34)
(135,2,4; 12,3,45)	(135,2,4; 14,25,3)	(135,2,4; 1,245,3)
(135,2,4; 1,25,34)	(13,25,4; 12,3,45)	(13,25,4; 145,2,3)
(13,25,4; 14,2,35)	(13,25,4; 15,24,3)	(13,25,4; 1,24,35)
(13,25,4; 15,2,34)	(13,25,4; 1,2,345)	(13,2,45; 125,3,4)
(13,2,45; 12,35,4)	(13,2,45; 14,25,3)	(13,2,45; 15,24,3)
(13,2,45; 1,24,35)	(13,2,45; 1,25,34)	(125,3,4; 14,2,35)
(125,3,4; 1,24,35)	(125,3,4; 1,2,345)	(12,35,4; 145,2,3)
(12,35,4; 14,25,3)	(12,35,4; 15,24,3)	(12,35,4; 1,245,3)
(12,35,4; 15,2,34)	(12,35,4; 1,25,34)	(12,3,45; 14,2,35)
(12,3,45; 1,24,35)	(12,3,45; 15,2,34)	(12,3,45; 1,25,34)
(145,2,3; 1,24,35)	(145,2,3; 1,25,34)	(14,25,3; 1,24,35)
(14,25,3; 15,2,34)	(14,25,3; 1,2,345)	(14,2,35; 15,24,3)
(14,2,35; 1,245,3)	(14,2,35; 1,25,34)	(15,24,3; 1,25,34)
(15,24,3; 1,2,345)	(1,245,3; 15,2,34)	(1,24,35; 15,2,34)

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On Signal Flow Graph Analysis of Ladder Networks

Abstract—A unified short-cut tool for the analysis of both network and sensitivity functions of a ladder network is given. New results on the sensitivity analysis provide a simple useful method for laboratory measurements.

I. INTRODUCTION

Special techniques for the analysis of a ladder network have received extensive attention in the past [1]–[5]. All have been mainly concerned with short-cut methods of obtaining network functions or their coefficients. One particularly attractive method seems to be the recurrence formula of Bashkow, who obtained the result based on a property of the triple diagonal matrix [2].

In this correspondence it is shown that the use of the signal flow graph leads to several recurrence formulas (including the one in [2]) by inspection. Then, as with Bashkow's formulas, network functions are obtained directly. Moreover, upon investigating some properties of the sequence of functions so generated, it is shown that all sensitivity functions can be formed readily from these functions. Physical interpretations of the results lead to new insight and provide simple measurement techniques in the sensitivity analysis of ladder networks.

II. THE GRAPH AND PROPERTIES OF ITS TRANSMISSIONS

For the ladder network in Fig. 1, one may write

$$\begin{aligned} V_n &= Z_{n-1}I_{n-1} + V_{n-2}, \qquad V_n = E \\ I_{n-1} &= Y_{n-2}V_{n-2} + I_{n-3} \\ &\cdots \\ V_2 &= Z_1 I_1 + V_0 \\ I_1 &= Y_0 V_0, \qquad I_{-1} = 0. \end{aligned} \tag{1}$$

A signal flow graph representation of (1) is given in Fig. 2 [1]. The graph contains no feedback loops. Let the node variables of the graph be $x_0, x_1, \cdots, x_n$, corresponding to $V_0, I_1, \cdots, V_n$, and let the transmission from x_0 to x_k be T_k. Then, it is immediately clear that

$$T_k = h_{k-1}T_{k-1} + T_{k-2}, \qquad k = 1, 2, \cdots, n \tag{2a}$$

$$T_0 \triangleq 1 \qquad T_{-1} \triangleq 0 \tag{2b}$$

where

$$h_0 = Y_0 \qquad h_1 = Z_1, \cdots, h_{n-1} = Z_{n-1}$$

since there are two incoming paths to node x_k (except x_1), with transmissions h_{k-1} and 1 from node x_{k-1} and x_{k-2}, respectively. Equation (2) has been obtained by Bashkow with an entirely different meaning for T_k; i.e., it is the determinant of a kth-order matrix. The graph gain from the source node x_0 to x_k is

$$\frac{x_k}{x_0} = T_k \tag{3a}$$

and in particular,

$$\frac{x_n}{x_0} = \frac{E}{x_0} = T_n. \tag{3b}$$

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