A COMBINATORIAL INTERPRETATION FOR THE BINOMIAL COEFFICIENTS

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We give a combinatorial interpretation for the binomial coefficients in terms of the descent statistic on a subset of the full symmetric group.

Let S_n denote the symmetric group of permutations of the set $[n] := \{1, ..., n\}$. If $\sigma = \sigma_1 ... \sigma_n$ is an element of S_n written in one line notation then we say σ has a descent at position i if $\sigma_i > \sigma_{i+1}$ and define $\mathrm{DES}(\sigma)$, the descent set of σ, by

$$\mathrm{DES}(\sigma) := \{i \in [n-1] : \sigma_i > \sigma_{i+1}\}.$$

A permutation statistic is a mapping from S_n into the set of nonnegative integers. We consider two permutation statistics, the descent number $\mathrm{des}(\sigma)$ defined as

$$\mathrm{des}(\sigma) :=\mid \mathrm{DES}(\sigma) \mid$$

and MacMahon's major index $\mathrm{maj}(\sigma)$ defined as

$$\mathrm{maj}(\sigma) := \sum_{i \in \mathrm{DES}(\sigma)} i.$$

It is well-known that the Eulerian numbers enumerate permutations in the symmetric group by descents, that is,

$$\sum_{\sigma \in S_n} t^{\mathrm{des}(\sigma)} = A_n(t),$$

where $A_n(t)$ dentes the n-th Eulerian polynomial of degree $n-1$ (see, for example, [1]).
MacMahon [2] found the closed-form for the generating polynomials of the major index

$$\sum_{\sigma \in S_n} q^{\mathrm{maj}(\sigma)} = 1(1+q)(1+q+q^2)\cdots(1+q+\cdots+q^{n-1}).$$

The purpose of this note is to find the corresponding results for the descent number and major index statistics when they are restricted to a particular subset of the full symmetric group.
Let $S(n)$ denote the subset of S_n consisting of those permutations σ of $[n]$ such that $\sigma_{i+1} - \sigma_i \leq 1$ for $i = 1, ..., n-1$. Thus the permutations in $S(n)$ are those that, when read from left to right, never increase by more than 1. Given a permutation σ in $S(n)$ we can construct a permutation in $S(n+1)$ in one of two ways: either place $n+1$ in front of σ or insert $n+1$ into σ immediately after the occurence of n. It is easy to see that we obtain all the permutations in $S(n+1)$ in this way. Consequently, $\mid S(n+1) \mid = 2^n$. The first few cases are easily listed:

$S(1)$	$S(2)$	$S(3)$	$S(4)$
1	12	123	1234
	21	231	2341
		312	3412
		321	3421
			4123
			4231
			4312
			4321

Let us examine the distribution of the descent number and the major index on, for example, $S(4)$.

σ	$\mathrm{des}(\sigma)$	$\mathrm{maj}(\sigma)$
1234	0	0
2341	1	3
3412	1	2
3421	2	5
4123	1	1
4231	2	4
4312	2	3
4321	3	6

From the table we see that the generating function for descents in $S(4)$ is

$$\sum_{\sigma \in S(4)} t^{\mathrm{des}(\sigma)} = 1 + 3t + 3t^2 + t^3 = (1+t)^3,$$

whilst for the major index we have

$$\begin{aligned} \sum_{\sigma \in S(4)} q^{\mathrm{maj}(\sigma)} &= 1 + q + q^2 + 2q^3 + q^4 + q^5 + q^6 \\ &= (1+q)(1+q^2)(1+q^3). \end{aligned}$$

The bivariate generating function for the pair of statistics (maj, des) also has the form of a product

$$\begin{aligned} \sum_{\sigma \in S(4)} q^{\mathrm{maj}(\sigma)} t^{\mathrm{des}(\sigma)} &= 1 + t(q + q^2 + q^3) + t^2(q^3 + q^4 + q^5) + t^3 q^6 \\ &= (1+qt)(1+q^2t)(1+q^3t). \end{aligned}$$

We shall show that the natural generalization of these results hold for all $S(n)$. Firstly, we consider the distribution of the descent statistic on $S(n)$.

Proposition 1. *The binomial coefficient* $\binom{n}{k}$ *counts the number of permutations in* $S(n+1)$ *with* k *descents. Equivalently,*

$$\sum_{\sigma \in S(n+1)} t^{\text{des}(\sigma)} = (1+t)^n.$$

Proof. Let $a(n,k)$ denote the number of permutations in $S(n)$ with k descents. If we place $n+1$ in front of a permutation $\sigma \in S(n)$ we get an element of $S(n+1)$ with one extra descent; if we insert $n+1$ into σ immediately after the occurrence of n we get an element of $S(n+1)$ having the same number of descents as σ. Since we obtain all the elements of $S(n+1)$ in one of these two ways we have the recurrence equation

$$a(n+1,k) = a(n,k-1) + a(n,k).$$

This is the same as the recurrence for the binomial coefficients but the initial condition is now $a(1,0) = 1$. Consequently,

$$a(n+1,k) = \binom{n}{k}.$$

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In order to investigate the joint distribution of the descent and major index statistics on $S(n)$ we introduce a coding for the permutations in $S(n)$ in terms of binary numbers.
Let BIN(n) denote the set of n bit binary numbers

$$\text{BIN}(n) = \{b_1...b_n : b_i \in \{0,1\}\}.$$

Definition. *Define a mapping* $\phi_{n+1} : S(n+1) \to BIN(n)$, *for* $n \geq 1$, *as follows: if* $\sigma = \sigma_1...\sigma_{n+1}$ *is an element of* $S(n+1)$ *put* $\phi_{n+1}(\sigma) = b_1...b_n \in BIN(n)$, *where*

$$b_i = \begin{cases} 1 & \text{if } \sigma \text{ has a descent at position } i \in [n] \\ 0 & \text{otherwise.} \end{cases}$$

For example, the table below shows the action of the map ϕ_4 on S(4). We see that ϕ_4 is a bijection from $S(4)$ onto the set of 3-bit binary numbers.

$\sigma \in S(4)$	$\phi_4(\sigma) \in \text{BIN}(3)$
1234	000
2341	001
3412	010
3421	011
4123	100
4231	101
4312	110
4321	111

One can quickly verify that the maps ϕ_2 and ϕ_3 are also bijections.

Claim. *For $n = 1, 2, ...$ the map $\phi_{n+1} : S(n+1) \to BIN(n)$ is a bijection.*

Sketch proof. A proof by induction can be given. Having established the initial cases up to $S(4)$, the idea behind the inductive step can be seen if we examine how the binary code associated with a permutation changes as we move from $S(4)$ to $S(5)$.

Notice that in the previous table the first nonzero bit (reading from the left) in each of the 3 bit binary codes is in the same position as the digit 4 in the corresponding permutation in $S(4)$. The permutations in $S(5)$ are obtained from those in $S(4)$ in one of two ways.

Firstly, we can place the digit 5 in front of each element of $S(4)$. This introduces an extra descent for each permutation in position 1 and thus an extra bit equal to 1 in front of the above 3 bit binary codes. The resulting 4 bit binary numbers represent the integers 8 through 15.

Alternatively, we can insert the digit 5 immediately after the occurrence of 4 in each element of $S(4)$. The permutation 1234 becomes 12345 with binary code 0000. In the other cases the insertion has the effect of replacing the leftmost nonzero bit in the 3 bit binary code with the two bit string 01. In all cases the result is an extra bit equal to 0 in front of the above 3 bit binary codes. The resulting 4 bit binary numbers represent the integers 0 through 7.

This shows that the map $\phi_5 : S(5) \to \mathrm{BIN}(4)$ is a bijection. Moreover, the first nonzero bit in each of the 4 bit binary codes is in the same position as the digit 5 in the corresponding permutation.

Clearly, we can continue in this manner and give an inductive proof that each mapping $\phi_{n+1} : S(n+1) \to \mathrm{BIN}(n)$, $n = 1, 2, ...$, is a bijection.
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Let the permutation $\sigma \in S(n+1)$ map to the binary number $\phi_{n+1}(\sigma) = b_1 ... b_n$. It follows from the definition of the map ϕ_{n+1} that the descent number and major index of σ are given by

$$\begin{aligned} \mathrm{des}(\sigma) &= \sum_{i=1}^{n} b_i \\ \mathrm{maj}(\sigma) &= \sum_{i=1}^{n} i b_i. \end{aligned}$$

Thus the bivariate generating function $G_{n+1}(q, t)$ for the joint distribution of the pair of statistics (maj, des) on the set of permutations $S(n+1)$

$$G_{n+1}(q, t) := \sum_{\sigma \in S(n+1)} q^{\mathrm{maj}(\sigma)} t^{\mathrm{des}(\sigma)}$$

can be written in the form

$$G_{n+1}(q,t) = \sum_{b_1\ldots b_n \in \mathrm{BIN}(n)} q^{\sum_{i=1}^{n} ib_i} t^{\sum_{i=1}^{n} b_i} .$$

Proposition 2.

$$G_{n+1}(q,t) := \sum_{\sigma\in S(n+1)} q^{\mathrm{maj}(\sigma)} t^{\mathrm{des}(\sigma)} = (1+qt)(1+q^2t)\cdots(1+q^nt) \quad [n = 1, 2, ...].$$

Proof. The proof is by induction on n. The initial case $n = 1$ is easy to verify. Assume then that for some positive integer n there holds

$$G_{n+1}(q,t) = (1+qt)(1+q^2t)\cdots(1+q^nt) \quad .$$

Then we have

$$\begin{aligned}
(1+qt)\cdots(1+q^{n+1}t) &= (1+q^{n+1}t)G_{n+1}(q,t) \\
&= (1+q^{n+1}t)\left\{ \sum_{b_1\ldots b_n \in \mathrm{BIN}(n)} q^{\sum_{i=1}^{n} ib_i} t^{\sum_{i=1}^{n} b_i} \right\} \\
&= \sum_{b_1\ldots b_n \in \mathrm{BIN}(n)} q^{\sum_{i=1}^{n} ib_i} t^{\sum_{i=1}^{n} b_i} + \sum_{b_1\ldots b_n \in \mathrm{BIN}(n)} q^{n+1+\sum_{i=1}^{n} ib_i} t^{1+\sum_{i=1}^{n} b_i} \\
&= \sum_{b_1\ldots b_{n+1} \in \mathrm{BIN}(n+1)} q^{\sum_{i=1}^{n+1} ib_i} t^{\sum_{i=1}^{n+1} b_i} \\
&= G_{n+2}(q,t),
\end{aligned}$$

where we have used the elementary fact that the set $\mathrm{BIN}(n+1)$ of $n+1$ bit binary numbers may be obtained from the set $\mathrm{BIN}(n)$ by either appending a 0 bit or a 1 bit to the end of each n bit binary number.
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REFERENCES

1. R. L. Graham, D. E. Knuth, O. Patashnik, *Concrete Mathematics: A Foundation for Computer Science,* 2nd ed. Reading, MA: Addison-Wesley, pp. 267-272, 1994.

2. P.A. MacMahon, *The indices of permutations and the derivation therefrom of functions of a single variable associated with the permutations of any assemblage of objects*, Amer. J. Math. 35 (1913), no. 3, 281–322.