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Ising Model

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[DG74]

6. Ising Model

C. Domb

Physics Department, King's College, University of London, England.

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I. Introduction

A. Historical survey

In 1925 Uhlenbeck and Goudsmit put forward the hypothesis that the electron possesses a spin $s = \frac{1}{2}$, and that in a magnetic field its direction is quantized so that it orients either parallel or antiparallel to the field. In the same year where w_{lx} is the *counting weight* of the associated configuration determined by the following rules. For a zero-field graph with vertices of degree $2q_1, 2q_2 \ldots 2q_v$

$$w_{lx} = \sum_{l=1}^{v} \frac{q_l(q_l - 1)}{2}; \tag{2.16}$$

hence only graphs with at least one vertex of degree four need be considered, and the vertices of degree 2 make no contribution. To take typical examples for a graph with one vertex of degree 4, $w_{lx} = 1$, for two vertices of degree 4, $w_{lx} = 2$, for one vertex of degree 6, $w_{lx} = 3$. For a magnetic graph with odd vertices of order $(2r_1 + 1)$ and $(2r_2 + 1)$.

$$w_{lx} = r_1 r_2. \tag{2.17}$$

As typical examples for a graph with two odd vertices of degree 3, $w_{lx} = 1$, for one vertex of degree 3 and one of degree 5, $w_{lx} = 2$.

Although Sykes was able to establish the above results rigorously, his formal proof was never published; an elegant proof was given subsequently by Nagle and Temperley (1968). The counting theorem has proved to be of great practical use in extending susceptibility series for the simple Ising model, and it was subsequently generalized by Stanley to the D-dimensional classical vector model (Stanley, 1967; see this volume, Chapter 7). Even though the method requires the use of disjoint lattice constants, this does not become a serious handicap until terms of quite high order. As a result the coefficients $a_r^{(2)}$ in (1.41) have now been calculated as far as $r = 16$ for the p.t. (plane triangular), $r = 21$ for the s.q. (simple quadratic), and $r = 32$ for the h.c. (plane honeycomb) lattice in two dimensions, and $r = 17$ for the s.c. (simple cubic), $r = 15$ for the b.c.c. (body centred cubic) and $r = 12$ for the f.c.c. (face centred cubic) lattices in three dimensions (Sykes *et al.*, 1972 a, b). For the diamond lattice a subsequent calculation by Sykes and Gaunt (1973) gives coefficients up to $r = 22$. These susceptibility coefficients are reproduced in Tables I and II.

For the p.t. and h.c. lattices a "star-triangle" type of transformation discovered by Fisher (1959b) (see Syozi Vol. 1, Chapter 7) provides a useful

TABLE I. Susceptibility coefficients $a_r^{(2)}$ for two-dimensional lattices (eqn. 1.41). (Data from Sykes *et al.*, 1972*a*).

r/lattice	p.t.	s.q.	h.c.	r/lattice	h.c.
1	6	4	3	22	1 348998
2	30	12	6	23	2 403840
3	138	36	12	24	4 299018
4	606	100	24	25	7 677840
5	2586	276	48	26	13 635630
6	10818	740	90	27	24 206220
7	44574	1972	168	28	43 092888
8	181542	5172	318	29	76 635984
9	732678	13492	600	30	135 698970
10	2 935218	34876	1098	31	240 199320
11	11 687202	89764	2004	32	426 144654
12	46 296210	229628	3696		
13	182 588850	585508	6792		
14	717 395262	1 486308	12270		
15	2809 372302	3 763460	22140		
16	10969 820358	9 497380	40224		
17	—	23 918708	72888		
18	—	60 080156	130650		
19	—	150 660388	234012		
20	—	377 009300	421176		
21	—	942 105604	756624		

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TABLE II. Susceptibility coefficients $a_r^{(2)}$ for three-dimensional lattices (eqn. 1.41). (Data from Sykes *et al.*, 1972b; Gaunt and Sykes, 1973.)

r/lattice	f.c.c.	b.c.c.	s.c.	d.
1	12	8	6	4
2	132	56	30	12
3	1404	392	150	36
4	14652	2648	726	108
5	151116	17864	3510	324
6	1 546332	118760	16710	948
7	15 734460	789032	79494	2772
8	159 425580	5 201048	375174	8076
9	1609 987708	34 268104	1 769686	23508
10	16215 457188	224 679864	8 306862	67980
11	162961 837500	1472 595144	38 975286	196548
12	1 634743 178420	9619 740648	182 265822	566820
13		62823 141192	852 063558	1 633956
14		409297 617672	3973 784886	4 697412
15		2 665987 056200	18527 532310	13 501492
16			86228 667894	38 742652
17			401225 391222	111 146820
18				318 390684
19				911 904996
20				2608 952940
21				7463 042916
22				21328 259716

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equations (2.22) remain valid for the new model provided that we replace w^a, w^b, w^c ... by w_a, w_b, w_c ... ($w_a = \tanh \beta J_a$ etc.). For example, the partition function for the θ topology is

$$\ln Z^I(\theta) = \ln(1 + w_a w_b + w_b w_c + w_c w_a) \tag{2.24}$$

which is denoted in an obvious shorthand notation by $\ln(1 + ab + bc + ca)$. Following the procedure of Chapter 1, Section IV.B4, (2.24) is expanded and only terms containing abc are retained as follows:

$$-a^2bc - b^2ca - c^2ab + (a^3b^2c + a^2b^3c + ab^2c^3 + ab^3c^2 + a^3bc^2 + a^2bc^3) + 2a^2b^2c^2 + \ldots . \tag{2.25}$$

Each of these terms can be described as a *bonding* of the θ topology, the coefficient of which represents the *weight* of the bonding; each bonding can then be re-interpreted in terms of the original model ($w_a \to w^a$ etc.). Thus each bonding weight makes its contribution to an appropriate λ-weight, and as a particular example the term $a^r b^s c^t$ contributes to $w_\lambda(\theta; r+s+t-3)$.

In Chapter 1, Section IV.B4 a brief description is also given of a new method of calculating the bonding weights (Domb, 1972b) using the device of replacing a single interaction J by a parallel pair of interactions J', J'' for "ladder" topologies, and making a suitably chosen interaction infinite for non-ladder topologies. This appreciably facilitates the calculation of individual weights and enables a number of general theorems to be enumerated.

The method of different interactions described above can readily be computerized and has been used to calculate the coefficients $a_r^{(0)}$ in (1.40) for the standard three-dimensional lattices. Results are reproduced in Table III as far as $r = 14$ for the f.c.c. lattice, $r = 16$ for the b.c.c. lattice, $r = 18$ for the s.c. lattice, and $r = 22$ for the *d.* lattice.

For the standard two-dimensional lattices exact formulae are available for $\ln Z^I$, and there is no difficulty in calculating a substantial number of terms of $a_r^{(0)}$. This calculation has been much facilitated by a recent development due to Guttmann and Joyce (1972) (following a suggestion of Sykes). One of these authors (Joyce, 1974) has been able to determine differential equations for U_0^I in (1.40) from which the coefficients u_r can readily be found by means of a recurrence relation. Table IV lists the first 20 non-zero coefficients u_r for the standard two-dimensional lattices.

To calculate the zero field susceptibility for any net G we can make use of

TABLE III. Zero field $\ln Z_0^I$ coefficients $a_r^{(0)}$ for three-dimensional lattices (eqn. 1.40). (Data from Sykes *et al.*, 1972c; Sykes (private communication).)

r/lattice	f.c.c.	b.c.c.	s.c.	d.
3	8	—	—	—
4	33	12	3	—
5	168	—	—	—
6	930	148	22	2
7	5664	—	—	—
8	37018$\frac{1}{2}$	2496	187$\frac{1}{2}$	3
9	254986$\frac{2}{3}$	—	—	—
10	1 827768	52168	1980	24
11	13 520328	—	—	—
12	102 807720	1 242078	24044	69
13	795 503400	—	—	—
14	6279 937374	32 262852	319170	486
15		—	—	—
16		892 367762	4 514757$\frac{3}{4}$	2087$\frac{1}{2}$
17			—	—
18			67 003469$\frac{1}{3}$	13678$\frac{2}{3}$
19				—
20				72420
21				—
22				466238

equation (1.12) which tells us that χ_0 is the sum of $\langle \sigma_i \sigma_j \rangle_0$ for all pairs i, j of the lattice. Each $\langle \sigma_i \sigma_j \rangle_0$ can be derived from a $\ln Z^I$ calculation for a net G' consisting of the original net G with an additional bond J' connecting i and j. We then have

$$\langle \sigma_i \sigma_j \rangle = \lim_{K' \to 0} \frac{\partial}{\partial K'} \ln Z^I(G') = \lim_{w' \to 0} \frac{\partial}{\partial w'} \ln Z^I(G'). \tag{2.26}$$

If we expand $\ln Z^I(G')$ in terms of bonded topologies as above, we need only take account of bondings with a single line along J'.

The graphs contributing to $\chi_0(G)$ can therefore be derived from stars by breaking the whole or part of any line joining two principal points. Examples of such graphs derived from a θ-topology are shown in Fig. 10. We denote all graphs of this type derived from a parent star s_i as $c_t(s_i)$. The $\kappa(s_i)$ can be calculated for the parent star s_i (with the dashed part corresponding to an interaction J') by one of the methods described above, A2908 Pφ383 722

TABLE IV. Zero field coefficients u_r for two-dimensional lattices (eqn. 1.40).

r/lattice	h.c.(u_{2n})	s.q.(u_{2n})	p.t.(u_n)
1	1.5	2	3
2	0	4	6
3	3	8	12
4	−3	24	24
5	15	84	54
6	−24	328	138
7	93	1 372	378
8	−180	6 024	1080
9	639	27 412	3186
10	−1368	128 228	9 642
11	4 653	613 160	29 784
12	−10 605	2 985 116	93 552
13	35 169	14 751 592	297 966
14	−83 664	73 825 416	960 294
15	272 835	373 488 764	3 126 408
16	−669 627	1 907 334 616	10 268 688
17	2 157 759	9 820 757 380	33 989 388
18	−5 423 280	50 934 592 820	113 277 582
19	17 319 837	265 877 371 160	379 833 906
20	−44 354 277	1 395 907 472 968	1 280 618 784

and expanded as a power series, the coefficient of w^l being denoted by $\sigma[c_t(s_i)]$. If σ is now expanded as a power series in w, the coefficients of powers of w can conveniently be referred to as the χ-weights of $c_t(s_i)$, the coefficient of w^{l+m} being the $(m+1)$th order χ-weight denoted by $w_\chi(c_t; m)$ (l is the number of lines in c_t). We then have as a parallel to (2.23)

$$a_r^{(2)} = \sum_{m \geq 0} (c_t(s_i); L)\, w_\chi(c_t; m) \qquad (l = r - m) \tag{2.27}$$

the sum being taken over all derived graphs c_t of r lines or less. From the discussion of the previous paragraph we see that the χ-weights can be obtained from the set of bonding weights of star topologies.

The above procedure can be generalized to deal with $a_r^{(4)}$ and higher derivatives.

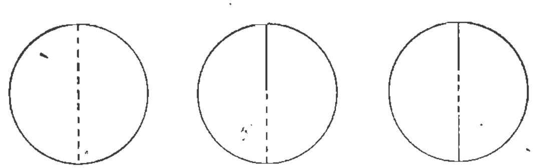

FIG. 10. Connected susceptibility graphs derived from a θ-topology.

For $a_r^{(4)}$ the contributing graphs can be derived from parent stars by breaking two sections of line joining principal points of the star. Again the appropriate weights can be obtained from the bonding weights of star topologies.

In practice, however, formulae like (2.27) have not been much used for calculating coefficients because of the difficulty of determining lattice constants of open graphs. For $a_r^{(2)}$ the method of the previous section has proved more useful, and for higher derivatives density expansions can be conveniently transformed as will be shown shortly. We shall later (Section II. B3) describe a procedure by which *all* series expansions for the simple Ising model can be expressed in terms of star lattice constants; this introduces complications in the weight calculations, but does offer the possibility of extending high temperature series expansions for the susceptibility and its derivatives.

4. *General spin s*

For spin s we use the Hamiltonian (1.3) and the partition function is given by (1.4) which we rewrite in the form

$$Z_{N,s}^{I} = \sum_{s_{zi}=-s}^{s} \prod_{\langle ij \rangle} \exp(4\bar{K} s_{zi} s_{zj}) \prod_{i} \exp(2\bar{L} s_{zi}) \tag{2.28}$$

where

$$\bar{K} = K/4s^2, \qquad \bar{L} = \beta m H/2s.$$

When $s = \frac{1}{2}$, $\bar{K}$ is identical with K, and $\bar{L} = \beta m H$. To apply the primitive method we expand each term of the first product as follows:

$$\exp(4\bar{K} s_{zi} s_{zj}) = 1 + 4\bar{K} s_{zi} s_{zj} + \frac{(4\bar{K})^2}{2!} (s_{zi} s_{zj})^2 + \frac{(4\bar{K})^r}{r!} (s_{zi} s_{zj})^r + \dots \tag{2.29}$$

The reduction of all the terms of (2.29) to single-bonded graphs is possible only for $s = \frac{1}{2}$, and for general s multiply-bonded graphs must be taken into account. Relation (2.2) can be generalized, and means that for spin s all bondings of order $(2s+1)$ or more can be eliminated. For lower values of s (particularly $s = 1$) this transformation might produce a simplification, but

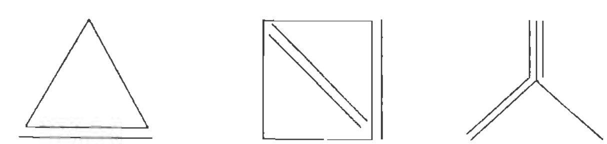

FIG. 11. Typical multiply-bonded graphs which enter into expansions for general s.

TABLE V. Zero-field $\ln Z_0{}^{\prime}$ and susceptibility coefficients for general spin s (eqns. 2.35 and 2.36). (Data from Domb and Sykes 1962.)

$$\hat{a}_2{}^{(0)}(s) = X^2/9$$
$$\hat{a}_3{}^{(0)}(s) = 8X^3/27$$
$$\hat{a}_4{}^{(0)}(s) = (X^2/225)\,(514X^2 - 116X + 1)$$
$$\hat{a}_5{}^{(0)}(s) = \left(\frac{48X^3}{405}\right)(184X^2 - 56X + 1)$$
$$\hat{a}_6{}^{(0)}(s) = (X^2/297675)\,(83599648X^4 - 36144288X^3 + 4664376X^2 - 118584X + 675)$$
$$\hat{a}_7{}^{(0)}(s) = (8X^3/14175)\,(7996592X^4 - 4275072X^3 + 817524X^2 - 35076X + 435)$$
$$\hat{a}_8{}^{(0)}(s) = (X^2/212625)\,(18568249616X^6 - 11735319488X^5 + 3100557664X^4 - 343347552X^3 + 14868306X^2 - 246780X + 945)$$
$$\hat{a}_0{}^{(2)}(s) = 1$$
$$\hat{a}_1{}^{(2)}(s) = 4X$$
$$\hat{a}_2{}^{(2)}(s) = (2X/5)\,(38X - 1)$$
$$\hat{a}_3{}^{(2)}(s) = (2X/75)\,(2124X^2 - 136X + 1)$$
$$\hat{a}_4{}^{(2)}(s) = (X/3150)\,(656648X^3 - 70772X^2 + 2322X - 15)$$
$$\hat{a}_5{}^{(2)}(s) = (X/330750)\,(251682608X^4 - 39096208X^3 + 2440236X^2 - 49104X + 225)$$
$$\hat{a}_6{}^{(2)}(s) = (X/1984500)\,(5480403392X^5 - 1125263472X^4 + 105206144X^3 - 4607196X^2 + 79290X - 315)$$

turned spin is $2mH + 2qJ$. However, for every bond between two overturned spins energy $2J$ is lost. Thus for a configuration of r overturned spins with m connecting bonds the energy of excitation is

$$2rmH + 2(qr - m)J\,, \tag{2.37}$$

and the corresponding Boltzmann factor is

$$(yz^q)^r z^{-2m} \qquad (z = u^{\frac{1}{2}} = \exp - 2\beta J)\,. \tag{2.38}$$

The number of excited energy states of this kind is the number of different configurations with r spins and m bonds that can be constructed from the net G. This is precisely the sum of strong lattice constants $[g_r; G]$ of all graphs g_r with r vertices and m bonds (see this volume Chapter 1, Section IV.A2). As in the case of high temperature expansions we can employ a primitive method which uses all lattice constants, or a cumulant method which uses only connected lattice constants.

1. [illegible]rimitive method

Using elementary counting procedures of the type described in an older

TABLE VI. Zero field $\ln Z_0{}^{\prime}$ and susceptibility coefficients for general spin s. Numerical values of higher coefficients† (eqns 2.35 and 2.36).

	$s = \frac{1}{2}$	exponent	$s = 1$	exponent	$s = 2$	exponent	$s = \infty$	exponent
$\hat{a}_9{}^{(0)}/X^9$	7.4379 55[illegible]96	05	1.3742 68026	06	1.7558 13494	06	1.9816 56984	06
$\hat{a}_{10}{}^{(0)}/X^{10}$	1.7732 73314	07	3.4442 32438	07	4.5138 62124	07	5.1651 89143	07
$\hat{a}_{11}{}^{(0)}/X^{11}$	4.7971 63396	08	9.7793 89933	08	1.3127 49781	09	1.5219 07693	09
$\hat{a}_{12}{}^{(0)}/X^{12}$	1.4521 50455	10	3.1039 34098	10	4.2632 38586	10	5.0045 41198	10
$\hat{a}_{13}{}^{(0)}/X^{13}$	4.8638 13992	11	1.0893 57606	12	1.5297 42374	12	1.8174 67603	12
$\hat{a}_7{}^{(2)}/X^7$	7.0798 48629	03	8.7743 92785	03	9.5604 80884	03	9.9790 38801	03
$\hat{a}_8{}^{(2)}/X^8$	2.3830 28081	04	3.0788 31315	04	3.4130 88269	04	3.5938 59837	04
$\hat{a}_9{}^{(2)}/X^9$	7.9944 21405	04	1.0770 68335	05	1.2150 60583	05	1.2908 46666	05
$\hat{a}_{10}{}^{(2)}/X^{10}$	2.6747 48676	05	3.7587 25793	05	4.3157 76041	05	4.6264 00504	05
$\hat{a}_{11}{}^{(2)}/X^{11}$	8.9294 79072	05	1.3090 72204	06	1.5300 29812	06	1.6551 08838	06
$\hat{a}_{12}{}^{(2)}/X^{12}$	2.9755 92746	06	4.5514 82885	06	5.4198 12170	06	5.9121 71481	06

†(Private communication from Professor M. Wortis. The final entries may contain a small systematic error but this should not affect exponent and amplitude analysis.)

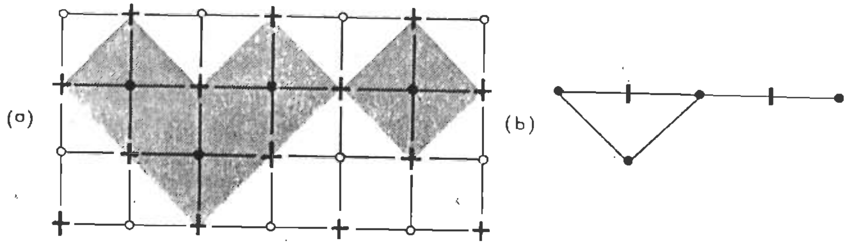

FIG. 18. (a) Simple quadratic with 4 spins overturned. (b) Corresponding shadow graph. O, B-spins; ●——●, 1st neighbour bond; ●, overturned B-spins; ●—▮—●, 2nd neighbour bond. (From Sykes *et al.*, 1973.)

second neighbour bonds, and a particular situation is illustrated in Fig. 18(a) and the corresponding shadow graph in Fig. 18(b). Since $q = 4$ the general code has 4 parameters. If r_1 and r_2 denote the number of first and second neighbour bonds in the shadow graph corresponding to $(\lambda,\alpha,\beta,\gamma,\delta)$ we find the relations (analogous to (2.119) and (2.120))

$$4s = \alpha + 2\beta + 3\gamma + 4\delta. \qquad (2.123)$$
$$2r_1 + r_2 = \beta + 3\gamma + 6\delta$$

Thus the s.q. codes can be used to derive the solution for the s.q. lattice with second neighbour interactions if the first energy is twice the second.

Analogous relations can be derived for the s.c. and b.c.c. lattices. However the need to introduce second and higher neighbour bonds greatly complicates the treatment.

2. *Numerical results. Series expansions*

Although the initial codes can be determined in an elementary way, complications rapidly increase and sophisticated techniques have been introduced to make further progress. These are described in more detail in a series of publications (Sykes *et al.*, 1973b, c, d, e); a computer programme for determining codes has been developed by Elliott (1969). We shall here confine our attention to a brief account of some of the important features.

In the first place we note that since the sub-lattices are symmetrical we must have

$$g_{st}(u) = g_{ts}(u). \qquad (2.124)$$

Thus any new code G_n must reproduce the sub-lattice polynomials g_{mn} correctly for all $m < n$. This *principle of complete code-balance* provides a check on the correctness of each new complete code as it is added. It implies a set of constraints on each complete code or partial generating function.

We have seen in the previous section that the codes depend on a limited number of parameters. Giving these parameters all possible numerical values we obtain the *algebraic* code system. However not all of these codes can be realized on the shadow lattice and it is convenient to distinguish between a *graphical* code which can be realized and a *non-graphical* code which cannot. To take a practical example for the h.c.–p.t. code β and γ are independent parameters and λ and α are then determined by (2.110) and (2.115). β and γ will be limited by the condition $\alpha \geqslant 0$ which gives

$$2\beta + 3\gamma \leqslant 3s. \qquad (2.125)$$

The number of distinct graphical codes in a complete code increases fairly slowly with s, and data from high temperature series expansions for the specific heat and susceptibility can be used to place constraints on these codes, using the transformation described in Section II.B2. By adding these to the symmetry constraints (2.124) it is possible to reduce very substantially the number of configurations to be counted. In practice a few extra configurations are counted to serve as a check.

TABLE VIII. Ferromagnetic polynomials $g_r(u)$ in a density expansion (eqn. 1.43).

p.t. lattice

$g_1 = u^3,$

$g_2 = 3u^5 - 3\frac{1}{2}u^6,$

$g_3 = 2u^6 + 9u^7 - 30u^8 + 19\frac{1}{3}u^9,$

$g_4 = 3u^7 + 12u^8 + 5u^9 - 178\frac{1}{2}u^{10} + 288u^{11} - 129\frac{3}{4}u^{12},$

$g_5 = 6u^8 + 21u^9 + 18u^{10} - 177u^{11} - 680u^{12} + 2637u^{13} - 2796u^{14} + 971\frac{1}{5}u^{15},$

$g_6 = 14u^9 + 42u^{10} + 33u^{11} - 278u^{12} - 1320u^{13} - 136\frac{1}{2}u^{14} + 16807u^{15} - 34920u^{16} + 27555u^{17} - 7796\frac{2}{3}u^{18},$

$g_7 = u^9 + 30u^{10} + 105u^{11} + 24u^{12} - 564u^{13} - 2682u^{14} - 3007u^{15} + 21168u^{16} + 63870u^{17} - 307476u^{18} + 437997u^{19} - 275184u^{20} + 65718\frac{1}{7}u^{21},$

$g_8 = 6u^{10} + 69u^{11} + 227u^{12} + 120u^{13} - 1822\frac{1}{2}u^{14} - 5313u^{15} - 8859u^{16} + 30825u^{17} + 165894\frac{1}{2}u^{18} - 58668u^{19} - 1907846\frac{1}{4}u^{20} + 4905025u^{21} - 5324130u^{22} + 2778678u^{23} - 574205\frac{7}{8}u^{24}.$

$g_9 = 27u^{11} + 160u^{12} + 483u^{13} + 228u^{14} - 4\,181u^{15} - 16\,704u^{16} - 11\,109u^{17} + 43\,868\frac{2}{3}u^{18} + 375\,483u^{19} + 408\,072u^{20} - 3\,019\,394u^{21} - 6\,438\,150u^{22} + 40\,681\,902u^{23} - 72\,302\,016u^{24} + 63\,438\,876u^{25} - 28\,314\,960u^{26} + 5\,157\,414\frac{4}{5}u^{27}$

$g_{10} = 3u^{11} + 86u^{12} + 432u^{13} + 837u^{14} + 449u^{15} - 10\,353u^{16} - 42\,315u^{17} - 48\,618\frac{1}{2}u^{18} + 205\,386u^{19} + 663\,288u^{20} + 1\,680\,030u^{21} - 4\,347\,964\frac{1}{2}u^{22} - 22\,703\,382u^{23} + 20\,150\,487u^{24} + 236\,013\,501\frac{3}{5}u^{25} - 741\,600\,943\frac{1}{2}u^{26} + 1\,012\,339\,456u^{27} - 745\,686\,690u^{28} + 290\,732\,760u^{29} - 47\,346\,449\frac{1}{5}u^{30}$

TABLE VIII *cont.*

s.q. lattice

$$
\begin{aligned}
g_1 &= u^2,\\
g_2 &= 2u^3 - 2\tfrac{1}{2}u^4,\\
g_3 &= 6u^4 - 16u^5 + 10\tfrac{1}{3}u^6,\\
g_4 &= u^4 + 18u^5 - 85u^6 + 118u^7 - 52\tfrac{1}{4}u^8,\\
g_5 &= 8u^5 + 43u^6 - 400u^7 + 926u^8 - 872u^9 + 295\tfrac{1}{5}u^{10},\\
g_6 &= 2u^5 + 40u^6 + 30u^7 - 1651u^8 + 5992\tfrac{2}{3}u^9 - 9144u^{10} + 6520u^{11} - 1789\tfrac{5}{6}u^{12},\\
g_7 &= 22u^6 + 136u^7 - 486u^8 - 5664u^9 + 33609u^{10} - 75640u^{11} + 85954u^{12}\\
&\quad - 49328u^{13} + 11397\tfrac{1}{7}u^{14},\\
g_8 &= 6u^6 + 134u^7 + 194\tfrac{1}{2}u^8 - 3986u^9 - 13323u^{10} + 164790u^{11} - 532196\tfrac{1}{2}u^{12}\\
&\quad + 867670u^{13} - 785091u^{14} + 377040u^{15} - 75238\tfrac{1}{8}u^{16},\\
g_9 &= u^6 + 72u^7 + 540u^8 - 1420u^9 - 19786u^{10} + 5112u^{11} + 691734u^{12}\\
&\quad - 3282328u^{13} + 7330033u^{14} - 9367653\tfrac{1}{3}u^{15} + 7040042u^{16}\\
&\quad - 2906956u^{17} + 510609\tfrac{4}{9}u^{18},\\
g_{10} &= 30u^7 + 461u^8 + 1144u^9 - 15480u^{10} - 66020u^{11} + 300885\tfrac{1}{2}u^{12}\\
&\quad + 2300266u^{13} - 17888832u^{14} + 53980742\tfrac{2}{5}u^{15} - 92320336u^{16}\\
&\quad + 97010462u^{17} - 62337864u^{18} + 22576512u^{19} - 3541971u^{20},\\
g_{11} &= 8u^7 + 310u^8 + 1864u^9 - 3373u^{10} - 91688u^{11} - 69358u^{12} + 2204652u^{13}\\
&\quad + 4259359u^{14} - 85259912u^{15} + 353290460u^{16} - 787713256u^{17}\\
&\quad + 1092475985u^{18} - 974679560u^{19} + 547000294u^{20} - 176425772u^{21}\\
&\quad + 25009987\tfrac{1}{11}u^{22},\\
g_{12} &= 2u^7 + 151u^8 + 1894u^9 + 3315u^{10} - 53428u^{11} - 383706\tfrac{2}{3}u^{12} + 1032758u^{13}\\
&\quad + 10552273u^{14} - 14665400u^{15} - 341367843\tfrac{1}{2}u^{16} + 2067415954u^{17}\\
&\quad - 5967607048\tfrac{1}{3}u^{18} + 10581976596u^{19} - 12347150173u^{20}\\
&\quad + 9570815133\tfrac{1}{3}u^{21} - 4767367976u^{22} + 1386008952u^{23}\\
&\quad - 179211452\tfrac{11}{12}u^{24},\\
g_{13} &= 68u^8 + 1340u^9 + 7389u^{10} - 20332u^{11} - 350828u^{12} - 965172u^{13}\\
&\quad + 10420351u^{14} + 32176924u^{15} - 210691538u^{16} - 1007111904u^{17}\\
&\quad + 10753093949u^{18} + 40670308548u^{19} + 90746211502u^{20}\\
&\quad - 133748320084u^{21} + 134710804372u^{22} - 92310171884u^{23}\\
&\quad + 41333506670u^{24} - 10938421828u^{25} + 1300139553\tfrac{1}{13}u^{26},\\
g_{14} &= 22u^8 + 864u^9 + 7\,372u^{10} + 11\,536u^{11} - 257\,378u^{12} - 1\,557\,816u^{13}\\
&\quad + 1\,314\,978u^{14} + 62\,452\,942u^{15} - 2\,072\,348u^{16} - 1\,354\,656\,284u^{17}\\
&\quad - 785\,938\,734u^{18} + 48\,542\,073\,472u^{19} - 250\,471\,809\,911\tfrac{1}{2}u^{20}\\
&\quad + 700\,726\,407\,966\tfrac{2}{7}u^{21} - 1\,278\,321\,358\,994u^{22} + 1\,613\,014\,033\,334u^{23}\\
&\quad - 1\,429\,269\,896\,596u^{24} + 877\,614\,310\,184u^{25} - 356\,891\,308\,190u^{28}\\
&\quad + 86\,670\,538\,138u^{27} - 9\,532\,294\,556\tfrac{6}{7}u^{28},\\
g_{15} &= 6u^8 + 456u^9 + 6\,404u^{10} + 24\,436u^{11} - 94\,888u^{12} - 1\,677\,728u^{13}\\
&\quad - 3\,997\,457u^{14} + 34\,493\,510\tfrac{2}{3}u^{15} + 267\,958\,908u^{16} - 885\,175\,436u^{17}\\
&\quad - 5\,903\,060\,870\tfrac{2}{3}u^{18} + 16\,408\,972\,700u^{19} + 177\,977\,336\,689\tfrac{1}{5}u^{20}\\
&\quad - 1\,388\,708\,571\,629\tfrac{1}{3}u^{21} + 4\,917\,742\,574\,549u^{22} - 10\,990\,712\,090\,268u^{23}\\
&\quad + 16\,983\,610\,970\,872\tfrac{2}{3}u^{24} - 18\,741\,629\,318\,887\tfrac{1}{5}u^{25}
\end{aligned}
$$

TABLE VIII *cont.*

s.q. lattice cont.

$$
\begin{aligned}
&\quad + 14\,825\,042\,097\,211u^{26} - 8\,245\,969\,418\,426\tfrac{2}{3}u^{27}\\
&\quad + 3\,071\,337\,551\,762u^{28} - 689\,136\,584\,016u^{29} + 70\,528\,002\,102\tfrac{2}{5}u^{30}.
\end{aligned}
$$

f.c.c. lattice

$$
\begin{aligned}
g_1 &= u^6,\\
g_2 &= 6u^{11} - 6\tfrac{1}{2}u^{12},\\
g_3 &= 8u^{15} + 42u^{16} - 120u^{17} + 70\tfrac{1}{3}u^{18},\\
g_4 &= 2u^{18} + 24u^{19} + 123u^{20} + 126u^{21} - 1653u^{22} + 2322u^{23} - 944\tfrac{1}{4}u^{24},\\
g_5 &= 30u^{22} + 96u^{23} + 448u^{24} + 792u^{25} - 2871u^{26} - 16296u^{27} + 49290u^{28}\\
&\quad - 45792u^{29} + 14303\tfrac{1}{5}u^{30},\\
g_6 &= u^{24} + 30u^{25} + 168u^{26} + 776u^{27} + 1212u^{28} + 3930u^{29} - 6904u^{30} - 65070u^{31}\\
&\quad - 64224u^{32} + 771272u^{33} - 1329240u^{34} + 922152u^{35} - 234103\tfrac{1}{6}u^{36},\\
g_7 &= 8u^{27} + 36u^{28} + 336u^{29} + 1\,350u^{30} + 3\,528u^{31} + 9\,036u^{32} - 1160u^{33}\\
&\quad + 1\,038u^{34} - 281\,400u^{35} - 622\,498u^{36} + 1\,503\,912u^{37}\\
&\quad + 8\,356\,041u^{38} - 28\,260\,664u^{39} + 34\,148\,478u^{40} - 18\,902\,160u^{41}\\
&\quad + 4\,044\,119\tfrac{1}{7}u^{42},\\
g_8 &= 28u^{30} + 96u^{31} + 786u^{32} + 2\,432u^{33} + 9\,804u^{34} + 19\,314u^{35} + 29\,146u^{36}\\
&\quad + 20\,550u^{37} - 322\,950u^{38} - 474\,806u^{39} - 4\,371\,355\tfrac{1}{2}u^{40} + 1\,944\,846u^{41}\\
&\quad + 40\,271\,875u^{42} + 32\,438\,508u^{43} - 452\,857\,765\tfrac{1}{2}u^{44} + 916\,579\,240u^{45}\\
&\quad - 853\,695\,741u^{46} + 393\,105\,420u^{47} - 72\,699\,427\tfrac{1}{8}u^{48}.
\end{aligned}
$$

b.c.c. lattice

$$
\begin{aligned}
g_1 &= u^4,\\
g_2 &= 4u^7 - 4\tfrac{1}{2}u^8,\\
g_3 &= 28u^{10} - 64u^{11} + 36\tfrac{1}{3}u^{12},\\
g_4 &= 12u^{12} + 204u^{13} - 798u^{14} + 948u^{15} - 366\tfrac{1}{4}u^{16},\\
g_5 &= 12u^{14} + 216u^{15} + 1262u^{16} - 9072u^{17} + 17592u^{18} - 14184u^{19} + 4174\tfrac{1}{5}u^{20},\\
g_6 &= 27u^{16} + 312u^{17} + 2368u^{18} + 4312u^{19} - 92992u^{20} + 275021\tfrac{1}{3}u^{21}\\
&\quad - 353640u^{22} + 216036u^{23} - 51444\tfrac{1}{2}u^{24},\\
g_7 &= 72u^{18} + 704u^{19} + 4404u^{20} + 17616u^{21} - 36348u^{22} - 833064u^{23}\\
&\quad + 3795726u^{24} - 7072736u^{25} + 6798900u^{26} - 3344712u^{27}\\
&\quad + 669438\tfrac{1}{7}u^{28},\\
g_8 &= 4u^{19} + 198u^{20} + 2016u^{21} + 10300u^{22} + 41352u^{23} + 55536u^{24} - 989076u^{25}\\
&\quad - 6007194u^{26} + 46866408u^{27} - 122039509u^{28} + 166096620u^{29}\\
&\quad - 127471458u^{30} + 52501716u^{31} - 9066913\tfrac{1}{8}u^{32},\\
g_9 &= 24u^{21} + 692u^{22} + 5816u^{23} + 30714u^{24} + 99648u^{25} + 226692u^{26}\\
&\quad - 887688u^{27} - 13103579u^{28} - 24522136u^{29} + 514861877\tfrac{1}{3}u^{30}\\
&\quad - 1874111776u^{31} + 3435605052u^{32} - 3684304933\tfrac{1}{3}u^{33}\\
&\quad + 2353070344u^{34} - 833603008u^{35} + 126632261\tfrac{1}{9}u^{36},
\end{aligned}
$$

TABLE VIII *cont.*

b.c.c. lattice cont.

$$
\begin{aligned}
g_{10} = {}& 156u^{23} + 2\,418u^{24} + 19\,568u^{25} + 89\,832u^{26} + 312\,984u^{27} + 534\,960u^{28} \\
& - 582\,528u^{29} - 21\,524\,820u^{30} - 122\,555\,960u^{31} + 184\,704\,162u^{32} \\
& + 4\,891\,550\,184u^{33} - 25\,940\,728\,064u^{34} + 62\,669\,293\,900\tfrac{4}{5}u^{35} \\
& - 88\,827\,538\,116u^{36} + 78\,607\,759\,128u^{37} - 42\,991\,931\,004u^{38} \\
& + 13\,362\,730\,248u^{39} - 1\,812\,137\,048\tfrac{9}{10}u^{40}, \\
g_{11} = {}& 12u^{24} + 800u^{25} + 9\,720u^{26} + 65\,112u^{27} + 302\,497u^{28} + 897\,848u^{29} \\
& + 1\,976\,484u^{30} - 2\,366\,032u^{31} - 34\,701\,994u^{32} - 284\,193\,600u^{33} \\
& - 704\,476\,488u^{34} + 6\,025\,344\,368u^{35} + 36\,918\,882\,951u^{36} \\
& - 323\,871\,127\,432u^{37} + 1\,029\,543\,128\,536u^{38} - 1\,871\,827\,463\,448u^{39} \\
& + 2\,164\,621\,975\,492u^{40} - 1\,630\,783\,111\,424u^{41} + 779\,883\,805\,680u^{42} \\
& - 215\,938\,102\,896u^{43} + 26\,449\,153\,814\tfrac{1}{11}u^{44}.
\end{aligned}
$$

s.c. lattice

$$
\begin{aligned}
g_1 = {}& u^3, \\
g_2 = {}& 3u^5 - 3\tfrac{1}{2}u^6, \\
g_3 = {}& 15u^7 - 36u^8 + 21\tfrac{1}{3}u^9, \\
g_4 = {}& 3u^8 + 83u^9 - 328\tfrac{1}{2}u^{10} + 405u^{11} - 162\tfrac{3}{4}u^{12}, \\
g_5 = {}& 48u^{10} + 426u^{11} - 2804u^{12} + 5532u^{13} - 4608u^{14} + 1406\tfrac{1}{5}u^{15}, \\
g_6 = {}& 18u^{11} + 496u^{12} + 1575u^{13} - 22144\tfrac{1}{2}u^{14} + 64574u^{15} - 84738u^{16} + \\
& + 53370u^{17} - 13150\tfrac{2}{3}u^{18}, \\
g_7 = {}& 8u^{12} + 378u^{13} + 3888u^{14} - 1360u^{15} - 157380u^{16} + 674652u^{17} \\
& - 1261904u^{18} + 1240035u^{19} - 628236u^{20} + 129919\tfrac{1}{7}u^{21}, \\
g_8 = {}& u^{12} + 306u^{14} + 4622u^{15} + 22396\tfrac{1}{2}u^{16} - 106113u^{17} - 947582\tfrac{1}{2}u^{18} \\
& + 6392769u^{19} - 16362155\tfrac{1}{4}u^{20} + 22521935u^{21} - 17686675\tfrac{1}{2}u^{22} \\
& + 7496787u^{23} - 1336290\tfrac{3}{8}u^{24}, \\
g_9 = {}& 24u^{14} + 127u^{15} + 5544u^{16} + 40050u^{17} + 60804u^{18} - 1368954u^{19} \\
& - 3978300u^{20} + 54753064u^{21} - 190517760u^{22} + 348702921u^{23} \\
& - 379686836u^{24} + 248294610u^{25} - 90480828u^{26} + 14175534\tfrac{1}{9}u^{27}, \\
g_{10} = {}& 24u^{15} + 396u^{16} + 4131u^{17} + 67267u^{18} + 236808u^{19} - 614784u^{20} \\
& - 12412763u^{21} + 2839656u^{22} + 414942978u^{23} - 2018275270u^{24} \\
& + 4793140380\tfrac{3}{5}u^{25} - 6835882485u^{26} + 6156900766u^{27} \\
& - 3449297064u^{28} + 1102444428u^{29} - 154094468\tfrac{7}{10}u^{30}, \\
g_{11} = {}& 24u^{16} + 660u^{17} + 6656u^{18} + 70275u^{19} + 602928u^{20} + 423644u^{21} \\
& - 12635748u^{22} - 86214999u^{23} + 306005260u^{24} + 2620578876u^{25} \\
& - 19491928200u^{26} + 59739201959u^{27} - 108143883564u^{28} \\
& + 126406988784u^{29} - 97076564452u^{30} + 47569139712u^{31} \\
& - 13540389348u^{32} + 1708597533\tfrac{1}{11}u^{33}, \\
g_{12} = {}& 3u^{16} + 1\,080u^{18} + 11\,562u^{19} + 101\,685u^{20} + 814\,709u^{21} + 3\,894\,597u^{22} \\
& - 12\,171\,177u^{23} - 135\,740\,953u^{24} - 397\,387\,542u^{25} + 4\,338\,189\,541\tfrac{1}{2}u^{26}
\end{aligned}
$$

TABLE VIII *cont.*

s.c. lattice cont.

$$
\begin{aligned}
& + 11\,093\,270\,424\tfrac{2}{3}u^{27} - 170\,115\,111\,953\tfrac{1}{4}u^{28} + 682\,270\,008\,351u^{29} \\
& - 1\,542\,754\,484\,221u^{30} + 2\,260\,372\,621\,941u^{31} - 2\,238\,908\,395\,410u^{32} \\
& + 1\,498\,634\,619\,771u^{33} - 652\,575\,075\,531u^{34} + 167\,442\,968\,667u^{35} \\
& - 19\,258\,135\,545u^{36} \\
g_{13} = {}& 96u^{18} + 732u^{19} + 23\,976u^{20} + 163\,820u^{21} + 1\,256\,172u^{22} + 6\,874\,170u^{23} \\
& + 12\,343\,160u^{24} - 220\,608\,330u^{25} - 1\,032\,194\,100u^{26} \\
& + 226\,958\,615u^{27} + 43\,210\,929\,384u^{28} - 18\,514\,105\,314u^{29} \\
& - 1\,306\,808\,581\,968u^{30} + 7\,163\,363\,995\,983u^{31} - 20\,147\,356\,102\,164u^{32} \\
& + 36\,242\,844\,825\,794u^{33} - 44\,637\,329\,262\,900u^{34} + 38\,365\,757\,618\,721u^{35} \\
& - 22\,758\,644\,334\,336u^{36} + 8\,917\,222\,503\,222u^{37} - 2\,082\,822\,677\,172u^{38} \\
& + 220\,080\,372\,439\tfrac{1}{13}u^{39}.
\end{aligned}
$$

As a result codes have been calculated for the following values of s and are tabulated in the publications referred to above (Sykes *et al.*, 1965, 1973b,d): h.c. ($s \leqslant 10$), s.q. ($s \leqslant 7$), d ($s \leqslant 8$), s.c. ($s \leqslant 6$), b.c.c. ($s \leqslant 5$). From these codes the general ferrimagnetic polynomials $g_{st}(u)$ can readily be obtained using (2.98) and (2.99); putting y_+ equal to y_- and summing for $s + t = m$ the ferromagnetic polynomials $g_m(u)$ are derived. Since they represent an important body of numerical data for the Ising model we reproduce the results for the p.t., s.q., f.c.c., b.c.c. and s.c. lattices in Table VIII. For other lattices data are available as follows: h.c. ($m \leqslant 21$), d. ($m \leqslant 17$).

If the expansion (2.97) is rearranged as a series in u, we can obtain the low temperature polynomials $f_r^{(a)}(y_+, y_-)$ of (1.55) for a ferrimagnet, and when $y_+ = y_-$ the polynomials $f_r(y)$ of (1.52) for a ferromagnet. However it is possible to extend these series by enumerating a limited number of *partial* codes for larger values of s. It is the class of a code and its successive ranks which are significant in these u-series. As a result of the calculations of Sykes *et al.* (1973 c, e) the following terms are available:

h.c. ($r \leqslant 16$), s.q. ($r \leqslant 11$), p.t. ($r \leqslant 16$), d. ($r \leqslant 15$), s.c. ($r \leqslant 20$),

b.c.c. ($r \leqslant 28$), f.c.c. ($r \leqslant 40$).

We reproduce in Tables IX–XII the coefficients in the expansion of $\ln Z_0{}^I$ in zero field, the spontaneous magnetization M_0, and the initial ferromagnetic and antiferromagnetic susceptibilities χ_0, $\chi_0{}^{(a)}$ as follows:

$$
\ln Z_0{}^I = -\frac{q}{8}\ln u + u^{q/2}\sum_{r=0}^{\infty} b_r{}^{(0)}u^r \qquad (2.126)
$$

$$M_0/m = 1 - 2u^{q/2} - u^{q-1} \sum_{r=1}^{\infty} b_r^{(1)} u^r \tag{2.127}$$

$$\chi_0 = 4\beta m^2 u^{q/2} \sum_{r=0}^{\infty} b_r^{(2)} u^r \tag{2.128}$$

$$\chi_0^{(a)} = 4\beta m^2 u^{q/2} \sum_{r=0}^{\infty} b_{ra}^{(2)} u^r. \tag{2.129}$$

.3 *Spin* $s > \frac{1}{2}$

The configurational problems which arise in deriving density or low temperature series expansions for spin $s > \frac{1}{2}$ are basically the same as those for $s = \frac{1}{2}$. Using the Hamiltonian (1.3), a ground state with all spins aligned in a magnetic field H, the ground state having energy

$$-N(\tfrac{1}{2}qJ + mH). \tag{2.130}$$

This is identical with the ground state energy for $s = \frac{1}{2}$, and results from the normalization we have chosen in (1.3) for which the maximum interaction between two parallel spins and the maximum interaction with an external field are independent of s.

We then consider excited states of overturned spins; however there is no longer a single state of an overturned spin but there are $2s$ such states, and for any configuration of excited states the state of each spin must be specified. The problem parallels a many component fluid. By analogy with (1.16) and (1.17) we can write

$$kT \ln Z_N{}^I = \tfrac{1}{2}qJ + mH + kT \ln \Lambda_N{}^I(y,u) \tag{2.131}$$

where now

$$y = \exp - (\beta mH/s), \qquad u = \exp - (\beta J/s^2) \quad . \tag{2.132}$$

We can also develop series expansions for $\ln \Lambda^I(s)$ analogous to (1.43),

$$\ln \Lambda^I(s) = \sum_{r=1}^{\infty} y^r g_r(u), \tag{2.133}$$

where $g_r(u)$ is a polynomial in u whose highest power is u^{rqs}.

The primitive method for obtaining $g_r(u)$ was used by Sykes (1956) but was not taken very far because of the complexity of the resulting series and the difficulty of assessing critical behaviour. With the advent of more sophisticated methods of analysis (Gaunt and Guttmann, this volume,

TABLE IX. *Zero field coefficients $b_r^{(0)}$ of $\ln Z^I$ (eqn. 2.126).

Lattice	d.	s.c.	b.c.c.	f.c.c.
$r = 0$	1	1	1	1
1	2	0	0	0
2	$3\frac{1}{2}$	3	0	0
3	6	$-3\frac{1}{2}$	4	0
4	$12\frac{1}{3}$	15	$-4\frac{1}{2}$	0
5	30	-33	0	6
6	$83\frac{3}{4}$	$104\frac{1}{3}$	28	$-6\frac{1}{2}$
7	$250\frac{2}{3}$	$-280\frac{1}{2}$	-64	0
8	$768\frac{1}{5}$	849	$48\frac{1}{3}$	0
9	2442	$-2461\frac{3}{4}$	204	8
10	$8009\frac{1}{6}$	7485	-786	42
11	26956	$-22534\frac{1}{2}$	1164	-120
12	$93140\frac{2}{5}$	$69393\frac{1}{5}$	$922\frac{3}{4}$	$72\frac{1}{3}$
13	$3\ 29258\frac{1}{7}$	$-2\ 13754\frac{1}{2}$	-8760	24
14		$6\ 66750$	20032	123
15		$-20\ 86734\frac{1}{6}$	-9164	126
16		$65\ 83341$	$-84215\frac{4}{5}$	-1623
17		$-208\ 52363\frac{1}{4}$	$2\ 94677\frac{1}{3}$	2418
18			$-3\ 78996$	$-495\frac{1}{4}$
19			$-5\ 69704$	822
20			$38\ 32961\frac{1}{2}$	-2703
21			$-79\ 41796$	-15512
22			$11\ 18118$	50538
23			$430\ 16052$	-41526
24			$-1335\ 95088\frac{6}{7}$	$8777\frac{1}{5}$
25				-61446
26				-54402
27				$7\ 72624$
28				$-13\ 17960$
29				$6\ 61848$
30				$-8\ 20665\frac{1}{6}$
31				$15\ 49408$
32				$80\ 84382$
33				$-285\ 89452$
34				$298\ 89394\frac{1}{2}$

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* Values of $b_r^{(0)}$ for the h.c., s.q. and p.t. lattices in two dimensions can be derived from the high temperature coefficients in Table IV by a suitable transformation (Domb 1960; Syozi Vol. 1 Chapter 7).

TABLE X. Spontaneous magnetization coefficients $b_r^{(1)}$ (eqn. 2.127).

Lattice	h.c.*	s.q.	p.t.
$r = 0$	6	8	12
1	18	34	− 2
2	54	152	78
3	168	714	− 24
4	534	3472	548
5	1732	17318	− 228
6	5706	88048	4050
7	19038	4 54378	− 2030
8	64176	23 73048	30960
9	2 18190	125 15634	− 17670
10	7 47180	665 51016	2 42402
11	25 74488	3563 45666	− 1 52520
12	89 18070	19194 53984	19 32000
13	310 36560	103927 92766	− 13 12844
14	1084 57488	565272 00992	156 12150
15	3803 90574		− 112 97052
16	13384 95492		

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Lattice	d.	s.c.	b.c.c.	f.c.c.
$r = 0$	8	12	16	24
1	26	− 14	− 18	− 26
2	80	90	0	0
3	268	− 192	168	0
4	944	792	− 384	48
5	3474	− 2148	314	252
6	13072	7716	1632	− 720
7	49672	− 23262	− 6264	438
8	1 91272	79512	9744	192
9	7 44500	− 2 52054	10014	984
10	29 24680	8 46628	− 86976	1008
11	115 96284	− 27 53520	2 05344	− 12924
12	463 64456	92 05800	− 80176	19536
13		− 303 71124	− 10 09338	− 3062
14		1015 85544	35 79568	8280
15		− 3380 95596	−45 75296	− 26694
16			− 83 01024	− 1 53536
17			540 12882	5 07948
18			− 1126 40896	− 4 06056
19			51 64464	79532
20			6948 45120	− 7 29912
21			− 21607 81086	− 6 31608
22				92 79376
23				− 157 71600
24				− 4 67336
25				− 109 35114
26				218 35524
27				1127 52684
28				− 4005 76168
29				4102 87368

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* Expansion Variable $z = u^{1/2}$

The following recurrence relations may be noted (Sykes, private communication)

h.c. $(n+4)b_n^{(1)} = 4(n+3)b_{n-1}^{(1)} - (n+2)b_{n-2}^{(1)} - 6n_{n-3}^{(1)} + nb_{n-4}^{(1)} - 4(n-1)b_{n-5}^{(1)} + (n-2)b_{n-6}^{(1)}$

s.q. $(n+3)b_n^{(1)} = 6(n+2)b_{n-1}^{(1)} - 4b_{n-2}^{(1)} - 6nb_{n-3}^{(1)} + (n-1)b_{n-4}^{(1)}$

p.t. $(n+5)b_n^{(1)} = 10(n+3)b_{n-2}^{(1)} - 6b_{n-3}^{(1)} - 9(n+1)b_{n-4}^{(1)}$

TABLE XI. Low temperature ferromagnetic susceptibility coefficients $b_r^{(2)}$ (eqn. 2.128).

Lattice	h.c.*	s.q.	p.t.	d.	s.c.	b.c.c.	f.c.c.
$r = 0$	1	1	1	1	1	1	1
1	6	8	0	8	0	0	0
2	27	60	12	44	12	0	0
3	122	416	4	208	− 14	16	0
4	516	2791	129	984	135	− 18	0
5	2148	18296	122	4584	− 276	0	24
6	8792	1 18016	1332	21314	1520	252	− 26
7	35622	7 52008	960	98292	− 4056	− 576	0
8	1 43079	47 46341	10919	4 48850	17778	519	0
9	5 70830	297 27472	11372	20 38968	− 54392	3264	72
10	22 64649		1 32900	92 20346	2 13522	− 12468	378
11	89 42436		1 26396	415 45564	− 7 00362	20568	− 1080
12	351 69616		12 99851	1867 96388	26 01674	26662	665
13	1378 39308		13 49784	8286 23100	−88 36812	− 2 15568	384
14					319 25046	5 28576	1968
15					− 1103 23056	− 1 64616	2016
16					3930 08712	− 30 14889	− 25698
17					− 13695 33048	108 94920	39552
18						− 137 96840	− 3872
19						− 299 09616	20880
20						1904 23962	− 65727
21						− 3997 39840	− 3 79072
22						− 227 68752	12 77646
23						28034 02560	− 9 86856
24						− 87430 64909	1 76978
25							− 21 63504
26							− 18 18996
27							278 71080
28							− 471 38844
29							207 89424
30							− 365 09652
31							770 55330
32							3930 46656
33							14029 34816
34							14038 43388

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TABLE XII. Low temperature antiferromagnetic susceptibility coefficients $b_{ra}^{(2)}$ (eqn. 2.129).

Lattice	*h.c.	s.q.	d.	s.c.	b.c.c.
$r = 0$	1	1	1	1	1
1	0	0	0	0	0
1	3	4	4	0	0
3	2	8	0	− 2	0
4	12	39	16	15	− 2
5	24	152	24	− 36	0
6	80	672	122	104	28
7	222	3016	348	− 312	− 64
8	687	13989	1266	1050	39
9	2096	66664	4464	− 3312	224
10	6585		16394	10734	− 884
11	20892		57932	− 34518	1368
12	67216		2 15916	1 13210	1350
13	2 18412		8 28348	− 3 70236	− 12272
14				12 20922	28752
15				− 40 28696	− 11944
16				133 64424	− 1 38873
17				− 444 09312	4 94184
18					− 6 40856
19					− 11 11568
20					73 63194
21					− 154 88224
22					11 98848
23					935 06112
24					− 2934 73869

* Expansion variable $z = u^{1/2}$

PØ386 >214
PØ387 >215
PØ388 >216
PØ389 >217
PØ390 >218

Chapter 4) interest in the problem revived. The shadow lattice method can be generalized (Sykes and Gaunt, 1973) and the following tabulations have how been made for a number of two- and three-dimensional lattices (Fox and Gaunt, 1972):

$s = 1$

h.c. ($r \leqslant 12$); s.q. ($r \leqslant 10$); p.t. ($r \leqslant 7$); d. ($r \leqslant 12$);

s.c. ($r \leqslant 10$); b.c.c. ($r \leqslant 10$); f.c.c. ($r \leqslant 7$)

$s = 3/2$

p.t. ($r \leqslant 7$); f.c.c. ($r \leqslant 7$).

III. Critical Behaviour

In the previous sections we have been concerned with deriving power series expansions for various thermodynamic properties of Ising systems. The coefficients in these expansions are exact but they are limited in number. We first quote a few general properties of power series.

Consider a function $f(z)$ defined by (Dienes, 1931)

$$f(z) = \sum_{n=0}^{\infty} a_n z^n. \tag{3.1}$$

Then if

$$\lim_{n\to\infty} |a_n|^{1/n} \tag{3.2}$$

exists and is equal to $1/z_c$ the series converges for $|z| < z_c$. We can then write

$$|a_n| \sim f(n)/z_c^n, \tag{3.3}$$

where

$$\lim_{n\to\infty} [f(n)]^{1/n} = 1 \tag{3.4}$$

There is always a singularity on the circle $z = z_c$. If all of the a_n are consistent in sign, then the dominant singularity lies on the positive real axis. (For example the series in Tables I and II.) Replacing z by $-z$ we see that if the a_n alternate regularly the dominant singularity lies on the negative real axis (e.g. Table X p.t. and s.c.). More irregular alternations indicate dominant singularites in the complex plane (e.g. Table X b.c.c. and f.c.c.). If a_n is real these must occur in complex pairs $(1/z_c) \exp \pm (i\sigma)$. For a single pair we should expect

$$a_n \sim \frac{f(n)}{z_c^n} \cos n\,\sigma. \tag{3.5}$$

If σ is a simple fraction, this gives rise to cyclic behaviour, otherwise it is more random.

If all the a_n are known *exactly* we can (in principle) continue the function analytically across the whole plane. Asymptotic values of a_n determine the behaviour near to the dominant singularity. Hence we see that series of terms consistent in sign are particularly useful since a numerical analysis of the a_n provides direct information about the singularity of physical interest, which must be on the positive real axis to correspond to a positive temperature. When the terms are not consistent in sign there is a dominant unphysical singularity which masks the behaviour of the singularity of physical interest. We may then use a transformation in the complex plane which