NUMERATOR POLYNOMIAL COEFFICIENT ARRAY
FOR THE CONVOLVED FIBONACCI SEQUENCE

G. E. BERGUM
South Dakota State University, Brookings, South Dakota 57006
and
V. E. HOGGATT, JR.
San Jose State University, San Jose, California 95192

1. INTRODUCTION

In [1], [2], and [3], Hoggatt and Bicknell discuss the numerator polynomial coefficient arrays associated with the row generating functions for the convolution arrays of the Catalan sequence and related sequences. In [4], Hoggatt and Bergum examine the irreducibility of the numerator polynomials associated with the row generating functions for the convolution arrays of the generalized Fibonacci sequence \( \{ H_n \}_{n=1}^{\infty} \) defined recursively by

\[
H_1 = 1, \quad H_2 = P, \quad H_n = H_{n-1} + H_{n-2}, \quad n \geq 3,
\]

where the characteristic \( P^2 - P - 1 \) is a prime. The coefficient array of the numerator polynomials is also examined. The purpose of this paper is to examine the numerator polynomials and coefficient array related to the row generating functions for the convolution array of the Fibonacci sequence. That is, we let \( P = 1 \).

2. THE FIBONACCI ARRAY

We first note that many of the results of this section could be obtained from [4] by letting \( P = 1 \).

The convolution array, written in rectangular form, for the Fibonacci sequence is

<table>
<thead>
<tr>
<th>1</th>
<th>1</th>
<th>1</th>
<th>1</th>
<th>1</th>
<th>1</th>
<th>...</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>7</td>
</tr>
<tr>
<td>2</td>
<td>5</td>
<td>9</td>
<td>14</td>
<td>20</td>
<td>27</td>
<td>35</td>
</tr>
<tr>
<td>3</td>
<td>10</td>
<td>22</td>
<td>40</td>
<td>65</td>
<td>98</td>
<td>140</td>
</tr>
<tr>
<td>5</td>
<td>20</td>
<td>51</td>
<td>105</td>
<td>190</td>
<td>315</td>
<td>490</td>
</tr>
<tr>
<td>8</td>
<td>38</td>
<td>111</td>
<td>256</td>
<td>511</td>
<td>924</td>
<td>1554</td>
</tr>
<tr>
<td>13</td>
<td>71</td>
<td>233</td>
<td>594</td>
<td>1295</td>
<td>2534</td>
<td>4578</td>
</tr>
<tr>
<td>21</td>
<td>130</td>
<td>474</td>
<td>1324</td>
<td>3130</td>
<td>6588</td>
<td>12,720</td>
</tr>
</tbody>
</table>

The generating function \( C_m(x) \) for the \( m^{th} \) column of the convolution array is given by

\[
C_m(x) = (1 - x - x^2)^{-m}
\]

and it is obvious that

\[
C_m(x) = (x + x^2) C_m(x) + C_{m-1}(x).
\]

Hence, if \( R_{n,m} \) is the element in the \( n^{th} \) row and \( m^{th} \) column of the convolution array then the rule of formation for the convolution array is
Paid

2 People
came

Bermuda

Can

3 Seats
Call L
NUMERATOR POLYNOMIAL COEFFICIENT ARRAY

(2.3) \[ R_{n,m} = R_{n-1,m} + R_{n-2,m} + R_{n,m-1} \]

which is representable pictorially by

\[ \begin{array}{c}
\begin{array}{c}
\vdots \\
\vdots \\
u \cdot x
\end{array}
\end{array} \]

where

(2.4) \[ x = u + v + w. \]

If \( R_m(x) \) is the generating function for the \( m^{th} \) row of the convolution array then we see by (2.3) and induction that

(2.5) \[ R_1(x) = \frac{1}{1-x} \]

(2.6) \[ R_2(x) = \frac{1}{(1-x)^2} \]

and

(2.7) \[ R_m(x) = \frac{N_{m-1}(x) + (1-x)N_{m-2}(x)}{(1-x)^m} = \frac{N_m(x)}{(1-x)^m}, \quad m \geq 3 \]

with \( N_m(x) \) a polynomial of degree

\[ \left[ m - 1 \right] \]

where \( \left[ \right] \) is the greatest integer function.

The first few numerator polynomials are found to be

\[ \begin{array}{l}
N_1(x) = 1 \\
N_2(x) = 1 \\
N_3(x) = 2 - x \\
N_4(x) = 3 - 2x \\
N_5(x) = 5 - 5x + x^2 \\
N_6(x) = 8 - 10x + 3x^2 \\
N_7(x) = 13 - 20x + 9x^2 - x^3 \\
N_8(x) = 21 - 38x + 22x^3 - 4x^3
\end{array} \]

Recording our results by writing the triangle of coefficients for these polynomials, we have Table 2

| Coefficients of Numerator Polynomials \( N_m(x) \) |
|---|---|---|
| 1 | 1 |
| 2 | -1 |
| 3 | -2 |
| 5 | -5 | 1 |
| 8 | -10 | 3 |
| 13 | -20 | 9 | -1 |
| 21 | -38 | 22 | -4 |

Examining Tables 1 and 2, it appears as if there exists a relationship between the rows of Table 2 and the right diagonals of Table 1. In fact, we shall now show that