Scan (9 pages)

F. R. Bernhardt &
Nuss

Emails, April-May 1994

add to all seqs
mentioned on p. 1
(eleven of them)
Doug Rogers may coauthor a note with me on this sequence. He put me on the right track with it several years ago.

Your entry doesn't go very far, and has an error in the last number. It should be 5387, not 5386.

I found out a way to take my table obtained with a TRUE BASIC program and fix it to send to you. It is appended.

But first, let me insert a comment on A3436. All the above ramble about arkons is really A6343, not A3436. Your \$N states Hamilton circuits on n-octahedron. I found this sequence as following rather simple problem. Put 2n dots on a circle. How many ways can they be paired so that the two members of a pair are NOT consecutive. I.e. count the spaced circular pairings.

Enumerate ARKONS for n-gon with r pairs of vertices identified. Found as A6343 in NJA Sloane's On-line Encyclopedia of Integer Sequences.

\[
K(n,r) = \begin{array}{c|c|c|c|c}
\text{n} & \text{r} & K(n,0) & K(n,1) & K(n,2) \\
\hline
3 & 0..0 & 1 & & \\
4 & 0..1 & 2 & 2 & \\
5 & 0..1 & 5 & 5 & \\
6 & 0..2 & 14 & 15 & 5 \\
7 & 0..2 & 42 & 49 & 21 \\
8 & 0..3 & 132 & 168 & 84 & 14 \\
9 & 0..3 & 429 & 594 & 336 & 84 \\
10 & 0..4 & 1430 & 2145 & 1350 & 420 & 42 \\
11 & 0..4 & 4862 & 7865 & 5445 & 1980 & 330 \\
12 & 0..5 & 16796 & 29172 & 22022 & 9075 & 1980 & 132 \\
13 & 0..5 & 58786 & 109174 & 89232 & 40898 & 10725 & 1287 \\
\end{array}
\]

\text{TOTAL= 1}  \\
\text{TOTAL= 4}  \\
\text{TOTAL= 10}  \\
\text{TOTAL= 34}  \\
\text{TOTAL= 112}  \\
\text{TOTAL= 398}  \\
\text{TOTAL= 1443}  \\
\text{TOTAL= 5387}  \\
\text{TOTAL= 20482}  \\
\text{TOTAL= 79177}  \\
\text{TOTAL= 310102}
I see I put A3436 instead of A6343 on the table
I just sent you.
That's what I get for doing an on-line lookup of both at once,
and trying to read my printout too quickly.

The second diagonal of my table (first diag. and first col. are
both Catalan numbers) is
2 5 21 84 330 1287 5005 ...
This is NOT in your book, BUT
leaving off the initial 2, it IS !!

I will have to look at Cayley, but perhaps the 2 is implicitly
present in his counting, or _should be_.
If so, HIS #1607 is misfiled.

I append my results on A1003, A7297

A1003 "Super Catalan"
I had a lot of errors trying to verify this sequence, but many
were mine. I start with series y = [0 1 1 1 1 45 197 ...]x .
(The constant coeff. is simply omitted, not zero)
I obtain the functional equation
y = x(1 + y)(1 + 2y) >> I might have said 2+y in error
whence
4xyy' + (3x-1)y' + 2yy + 3y + 1 = 0
Then manipulating these equations, and using linear algebra I finally obtain (after removing many dumb errors)

\[ (-2x) + (-3x+1)y + x(xx - 6x + 1)y' = 0 \]

Now I have \( b(3) = 1, b(4) = 3, b(5) = 11, \ldots \) where \( b(n) \) is the number of polyoids: \( n \)-gon with any no. of diagonals. Thus

\[ (n+1) b(n+2) = 3(2n-1) b(n+1) - (n-2)b(n) \]

This shows that your recurrence had two errors: put \( n+1 \) not \( n-1 \) on LHS. Also your \( a(n) \) is my \( b(n+1) \).

I ALSO NOTICED SOMETHING WHICH TOTALLY SURPRISED ME.
Elsewhere I had looked at Schroder numbers, the subject of a problem in the monthly (AMM). Put a marker at the origin of the integer lattice. The pos. x-axis and pos. y-axis point E and N by the compass, respectively. A \_move_ consists of a step in the direction E, N, or NE (up, right, or diagonally up right).

How many paths which stay above \( y=x \) take you to \( (n,n) \)?

For \( n = 0,1,2,\ldots \) 1 2 6 22 90 394 1806 \ldots 
Note every value is even but first one.
The problem asked for demonstration that \( s(2), s(4), s(6), \ldots \) were divisible by 3.

Here I find that this seq. is (except for \( s(0) \)) the double of A1003 !! Hence \( b(4), b(6), b(8), \ldots \) are divisible by three also.

I will have to investigate combinatorial connections. The functional equation for Schro*der is \( y = x(1+y)(2+y) \) which is an easy transform from \( y = x(1+y)(1+2y) \) for polyoids, thus establishing the connection claimed.

Reference on Schroder Numbers:
Problem E3343. Proposed by Lou Shapiro and Doug Rogers

My response rated an "also solved" but I obtained a wider variety of interesting results than what they used. The main result was proved both by functional techniques and by pure combinatorics.

I found \( s(n) \) through \( s(18) = 600318853926 \).
Moreover,
\[ n S(n) \text{ is the coeff. of } t^{(n-1)} \text{ in } (2 + 3t + tt)^n . \]

\[
S(n) = \sum_{m \leq \sqrt{n}} \sum_{\substack{m+n \leq n \leq m+1 \leq r \leq m/n \leq \sqrt{n} \leq \sqrt{r-1} \leq n}} \frac{1}{m^2} = \sum_{n \leq m \leq n-m} \sum_{n+m \leq r \leq m+1} \frac{1}{n^2} \]

Reference:
C. DOMB, A.J. BARRETT, "Enumeration of Ladder Graphs"
I do not recommend this paper -- the terminology is awkward, and so is the lengthy derivation of asymptotic results. Precise formulae are relatively easy by Lagrange Inversion. They do define three enumeration problems \( a(n) \), \( b(n) \), \( c(n) \).

In my terminology:

\[
\begin{align*}
a(n) &= \sum_k a(n,k) : \text{the tabloids with } n \text{ vertices on a circle}, \\
&\quad \text{and } k \text{ non-crossing edges.}
\end{align*}
\]

\[
\begin{align*}
b(n) &= \sum_k b(n,k) : \text{the polyoids with } k \text{ diagonals in an } n\text{-gon.}
\end{align*}
\]

For Domb/B, the outer polygon edges are stripped off.

\[
\begin{align*}
c(n) &= \sum_k c(n,k) : \text{the count of } a(n,k) \text{ restricted to connected graphs.}
\end{align*}
\]

I gather these results and other similar ones in a paper close to completion: RINGED PLANAR GRAPHS Polyoid sequence \( b(n) \) is the seq. above linked to Schro"der.

Is there any general theory you know of that would connect identities belonging to a series \([0 \ 1 \ b \ c \ d \ldots]\) and its reversion \([0 \ 1 \ -b \ c' \ldots]\) ??

We still have the unexplained relation between \( c(n) \) of Domb/Barrett, namely \([x \ x \ 1 \ 4 \ 23 \ 156 \ 1162 \ldots]\) and the reversion of square sequence \([0 \ 1 \ 4 \ 9 \ 16 \ 25 \ldots]\) namely \([0 \ 1 \ -4 \ 23 \ -156 \ 1162 \ldots]\).

Does not this remarkable connection suggest the connected tabloids are more significant than so far appears, and cry out for more insight??

From njas Wed Apr 27 18:29 EDT 1994

about arkons, the seq. A6343:

you say place 2n points on a circle etc. now i just cannot get this to give the numbers \( a_3=1, a_4=4, a_5=10, \ldots \)

maybe you could explain a bit more? what are the "pairings" for \( n=3, n=4 \)?

you criticized the last entry in A6363, but didn't i get it from you?!

>From cs.rit.edu!frb6006 Thu Apr 28 17:00:58 EDT 1994

Subject: sequences, reply & update

I apologize that I managed too confuse two sequences A3436 A6343 (NOTE 343 + 6 in two ways!!!) I thought I had corrected this error last post, but I prob. wasn't clear enuf.

A3436
On-line Enc. says (%R) JCT B19 2 75 (%N) Hamilton circuits on n-c octah.
I created the sequence one day to answer the question, "How many ways can 2n dots on a circle be paired, if the two members of a pair may not be consecutive?"

For 2n= 4, one way: both diagonals
For 2n= 6, 4=1+3 : ‘asterisk’ & ‘double dagger’ x 3

It is fascinating that you list this seq. for something quite different! My issues of JCT (B) are stored at a friends, but soon I will go by a Univ. library and see what I can find out. But I think my description "spaced-circular pairs" is simpler.

A6343 (ARKONS)
\[
\text{sum}(r) \quad 1/ \quad \text{n} \quad / \quad 2n - 3r - 4 \quad / \\
\text{n} - r - 1 \quad \text{r} \quad / \quad \text{n} - 2r - 2 \\
\]

(R) TAMS 60 355 46
GD Birkhoff and DC Lewis compile values of Arkons = Elem. maps by brute force.

"Six-rings in Minimal Five-Color Maps" Arthur Bernhart
On p. 396+ he constructs 38 primary options
these are 32 of 34 elementary maps
(two omitted somehow, but it doesn’t affect the results)
and 6 not quite elementary added.

I find I really ought to survey the papers and the work of grad. students B.L. Poote, Slaughter (U of Okla) to summarize who has worked on this. So hold on this...

The error of 5386 for 5387 may well be my slip.
I may have sent you the computer-produced table for the above
K(n,r) and sum K(n), if so my apologies in advance for including it below.

The description: take a convex n-gon labeled 1,2,3,... at corners.
Preserving planarity and avoiding loops, pairs of non-consec.
vertices are merged through the interior.

We now have a celluloid, a tree-like plane rooted graph, every
block is a circuit (lunes allowed) of size 2,3,...
The celluloids A5043 are the companion to the Motzkin seq.

To complete the process, diagonals are added to cells (lunes &
triangles need none) to triangulate. Now we have ARKONS.
n=4: square, add either diagonal, or merge either diag. pair
n=5: triangulate (5 ways) or merge a pair (five ways)
The n=6 examples are the first 34 in the table 2, p. 185,
of K Appel, W Haken: _Every Planar Map is Four Colorable_,
This is probably a good reference to use.

Finally, the "brambles" are my name for the planar 2,3-cactii,
\( n = \# \text{ edges} \) make from lunes and triangles.
Note that this class is the intersection of celluloids and arkons.
These make a new sequence.

---

Enumerate ARKONS for \( n \)-gon with \( r \) pairs of vertices identified.
Found as A6343 in NJA Sloane's On-line Encyclopedia of Integer Sequences

\[
K(n,r) = \left\lfloor \begin{array}{l}
\frac{1}{n} \backslash \frac{2n - 3r - 4}{n-r-1} \backslash \frac{n - 2r - 2}{\ \ n} \end{array} \right\rfloor
\]

\[K(n) = \frac{n}{n-r-1} \frac{n-2r-2}{\ n} K(n,0) + K(n,1) + \ldots\]

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<th>( K(n,1) )</th>
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**Enumeration of Celluloids**

for \( n \)-gon, with \( r \) identified vertex pairs

\[R(n) = \text{gamma}(n) = \text{companion to Motzkin seq (studied by Riordan)}\]
\[= \text{linear dimension of } n\text{-ring chromatic polynomials}\]
\[\text{Motzkin}(n) = R(n) + R(n+1)\]

\[
\frac{1}{n} \frac{n-r-2}{r} \frac{n}{r+1}
\]

\[R(n,r) = \frac{\ \ r}{\ \ r} \frac{\ \ r}{\ \ r+1}\]
<table>
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<th>n</th>
<th>r</th>
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<th>TOTAL= R(n)</th>
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\[ \Delta = A_{132081} \]

Count Brambles: rooted planar cacti(2,3)
\[ n = \text{number of sides (2..20)} \]
\[ r = \text{number of blocks} \quad n/3 \leq r \leq n/2 \]

<table>
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<tr>
<th>( n )</th>
<th>( r )</th>
<th>( B(n,r) )</th>
<th>( B(n) = \text{TOTAL} )</th>
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In last column of line \( n=10 \), the figure 222 is missing.

From cs.rit.edu!frb6006 Fri May 6 19:31:41 EDT 1994

Wonderful! Thanks for the notice of new seq. analyzer. While I am considering what to try with it, a philosophical question occurs. Does SS (superseeker) halt when a match is found? More specifically, could it/has it looked for hidden relationships between sequences already catalogued.

If not, I can imagine a match with A#### leading to just one interpretation of the seq. of nominal importance, whereas if A#### had been missing from the catalog, then SS wd. keep trying and find much more meaningful relationship with ...

I went to the Graph Theory Day #27 near Morristown NJ last weekend. Half hoped I would see you, or Nate Dean, or other people from Bell research, not just Stefan Burr. No such luck...

Before I went, I went to Un. Roch. Carlson Library (Math, phys, chem..) and found additional uses of some sequences.

The minor discovery was the problem of Singmaster about seating \( n \) couples at a circular table so that no couple is side-by-side. This == his Hamilton circuits (labeled) on complement of complete \( n \)-partite \( K(2,2,\ldots,2) == n \)-icosahedron.

My sequence was the number of seating patterns: you only care where the partner of any position is located. I.e. divide out factors which distinguish men from women, or one couple from another.
I will
The companion to the Motzkin numbers turns out to appear in isomers
of tensors in chemistry, and in invariants in Hilbert algebras, and
in interval graphs.

I will finish this discussion (and give references) later;
there is static on the line right now.