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# FURTHER RESULTS ON LEFTIST TREES

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We deal with the class of leftist trees with $n$ leaves. Assuming that all these trees are equally likely to occur, asymptotic equivalents to some familiar parameters will be derived. These parameters are special cases of an arbitrary weight introduced in this paper. The presented results also imply that leftist trees are not simply generated.

## . Introduction

Let $T=(I,L,r)$ be an *extended binary tree* ([6;p.399]) with the set of internal nodes $I$, the set of leaves $L$ and the root $r\in I$. If $d(u,v)$ denotes the *distance* from node $u$ to node $v$ (=the number of nodes appearing in the shortest path from $u$ to $v$ minus 1) then *left branch length* $LBL(T)$ of $T$ is defined by $LBL(T):=d(r,a)$, where $a\in L$ is the leftmost leaf of $T$. The tree $T$ is said to be a *leftist tree* if

$$(\forall\, x\in I\cup L)(MIN\{d(x,v)\mid v\in L_x\}=LBL(T_x)),$$

where $T_x=(I_x,L_x,x)$ denotes the subtree of $T$ with root $x\in I\cup L$. The tree $T$ is called a *leftist tree of type* $\lambda\in\mathrm{N}$ if $T$ is a leftist tree with the property $LBL(T)=\lambda-1$.

Leftist trees play an important part in particular sorting algorithms based on selection; they involve an efficient way to represent priority queues as linked binary trees ([7;p.150–152,159]). The leftist trees with less than six leaves are drawn in Figure 1.

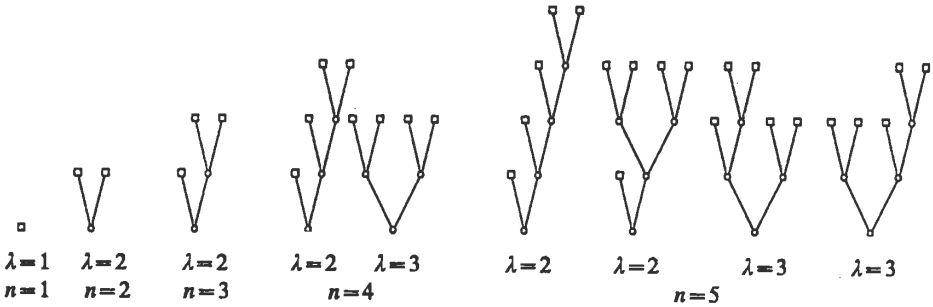


Figure 1. All leftist trees of type $\lambda$ with less than six leaves.

Now, let $t(n)$ ($t_\lambda(n)$, $\lambda \in \mathbb{N}$) be the number of all leftist trees (of type $\lambda$) with $n$ leaves. For a long time, the exact asymptotic behaviour of $t(n)$ was unknown ([7; p. 159]). But recently, the author succeeded in proving that ([2])

$$t(n) \sim \alpha \eta^{-n} n^{-3/2}, \tag{1}$$

where $\alpha = 0.250\,363\,429\,392...$ and $\eta = 0.363\,704\,091\,587...$ are constants.

The approach to the derivation of this result also involves a detailed analysis of some familiar parameters in the case of random leftist trees; when speaking of random leftist trees or random extended binary trees, we consider the situation where all possible leftist trees or extended binary trees with the same number of leaves are equally likely to occur. Thus, in this paper we present asymptotic equivalents to the expected values of the following parameters of a random leftist tree $T$:

- the left (right) branch length $LBL(T)$ ($RBL(T)$);
- the external (internal) path length $EPL(T)$ ($IPL(T)$);
- the external (internal; internal-external) free path length $EFPL(T)$ ($IRPL(T)$; $IEFPL(T)$);
- the left (right) path length $LPL(T)$ ($RPL(T)$);
- the number of external (internal; internal-external) root-free paths $EP_r(T)$ ($IP_r(T)$; $IEP_r(T)$);
- the number of leaves $L_l(T)$ ($L_r(T)$) appearing in the left (right) subtree of the root.

The formal definition of each of these quantities is presented in the second column of Table 1. Since for each extended binary tree with $n$ leaves the relations (cf. [4; p. 40])

- $IPL(T)=EPL(T)-2(n-1)$ (2a)
- $IFPL(T)=EFPL(T)+EPL(T)-2n(n-1)$ (2b)
- $IEFPL(T)=2EFPL(T)+EPL(T)-2n(n-1)$ (2c)
- $IP_r(T)=EP_r(T)-n+2$ (2d)
- $IEP_r(T)=2EP_r(T)$ (2e)

are valid, it is sufficient to investigate the remaining parameters. For this purpose, we introduce the following "additive weight" of a leftist tree:

Let $c_1, c_2 \in \mathrm{R}$ be two constants and let $g: \mathrm{N}_0^2 \to \mathrm{R}$, $\Phi_i: \mathrm{N}_0 \to \mathrm{R}$, $i \in \{1,2\}$, be given mappings. The weight $w(T)$ of a leftist tree $T=(I,L,r)$ of type $\lambda$ with the left subtree $T_1=(I_1,L_1,r_1)$ and the right subtree $T_2=(I_2,L_2,r_2)$ is recursively defined by

$$w(T) := \text{IF } |I \cup L| = 1 \text{ THEN } g(1,1)$$
$$\text{ELSE } g(\lambda, |L|) + c_1 w(T_1) + c_2 w(T_2)$$
$$+ \Phi_1(|L_1|) + \Phi_2(|L_2|);$$

All parameters (except $RPL(T)$) defined above can be characterized in a recursive way by choosing special values for $c_i$, $i \in \{1,2\}$, and special functions for $g$, $\Phi_1$ and $\Phi_2$. The particular choice of these quantities for each parameter is given in Table 1.

Table 1

The definition of distinguished parameters together with their representation as an additive weight. Here, $T=(I,L,r)$ is a leftist tree of type $\lambda$ with the left subtree $T_1=(I_1,L_1,r_1)$ and the right subtree $T_2=(I_2,L_2,r_2)$. The leftmost (rightmost) leaf of the subtree with root $v$ is denoted by $a_v(b_v)$.

| Parameter | Definition | Characterized by the weight $w(T)$ with | | | | |
|---|---|---|---|---|---|---|
| | | $c_1$ | $c_2$ | $g(\lambda,n)$ | $\Phi_1(n)$ | $\Phi_2(n)$ |
| $LBL(T)$ | $d(r,a_r)$ | 1 | 0 | $1-\delta_{\lambda,1}$ | 0 | 0 |
| $RBL(T)$ | $d(r,b_r)$ | 0 | 1 | $1-\delta_{\lambda,1}$ | 0 | 0 |

Table 1 contd.

| | | | | | | |
|---|---|---|---|---|---|---|
| $EPL(T)$ | $\sum_{v\in L} d(r,v)$ | 1 | 1 | 0 | $n$ | $n$ |
| $IPL(T)$ | $\sum_{v\in I} d(r,v)$ | 1 | 1 | 0 | $n-1$ | $n-1$ |
| $EFPL(T)$[1] | $\frac{1}{2}\sum_{(u,v)\in L\times L} d(u,v)$ | 1 | 1 | 0 | $n^2$ | $n^2$ |
| $IFPL(T)$[2] | $\frac{1}{2}\sum_{(u,v)\in I\times I} d(u,v)$ | 1 | 1 | 0 | $(n-1)^2$ | $(n-1)^2$ |
| $IEFPL(T)$[3] | $\sum_{(u,v)\in I\times L} d(u,v)$ | 1 | 1 | 0 | $2n(n-1)$ | $2n(n-1)$ |
| $LPL(T)$ | $\sum_{v\in I} d(v,a_v)$ | 1 | 1 | $\lambda-1$ | 0 | 0 |
| $EP_r(T)$ | $\binom{\lvert L_1\rvert}{2}+\binom{\lvert L_2\rvert}{2}$ | 0 | 0 | 0 | $\binom{n}{2}$ | $\binom{n}{2}$ |
| $IP_r(T)$ | $\binom{\lvert I_1\rvert}{2}+\binom{\lvert I_2\rvert}{2}$ | 0 | 0 | 0 | $\binom{n-1}{2}$ | $\binom{n-1}{2}$ |
| $IEP_r(T)$ | $\lvert I_1\rvert\lvert L_1\rvert+\lvert I_2\rvert\lvert L_2\rvert$ | 0 | 0 | 0 | $n(n-1)$ | $n(n-1)$ |
| $L_l(T)$ | $\lvert L_1\rvert$ | 0 | 0 | 0 | $n$ | 0 |
| $L_r(T)$ | $\lvert L_2\rvert$ | 0 | 0 | 0 | 0 | $n$ |

The remainder of this introduction is devoted to some basic results which we shall use in the subsequent sections.

As shown in [2], a leftist tree of type $(\lambda+1)$, $\lambda\in\mathrm{N}$, has the structure drawn in Figure 2, where the subtree $T_1$ is a leftist tree of type $\lambda$ and the subtree $T_2$ is a leftist tree of type $r\geqslant\lambda$. Furthermore, there is exactly one leftist tree of type 1, namely the one-node tree.

[1] $EFPL(T)$ and $w(T)$ are interrelated by $w(T)=|L|EPL(T)-EFPL(T)$ (see [4]).

[2] $IFPL(T)$ and $w(T)$ are interrelated by $w(T)=|I|IPL(T)-IFPL(T)$ (see [4]).

[3] $IEFPL(T)$ and $w(T)$ are interrelated by $w(T)=|I|\,EPL(T)+|L|IPL(T)-IEFPL(T)$ (see [4]).

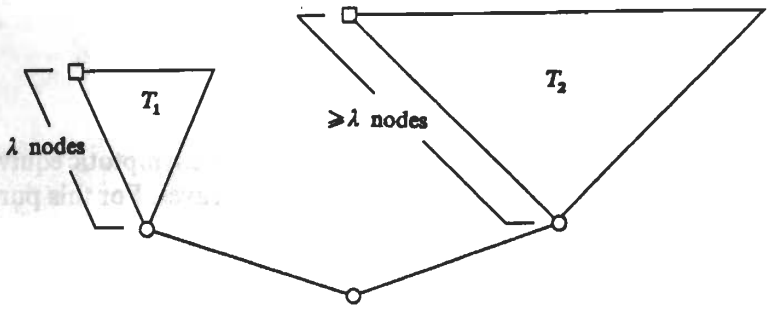


Figure 2. The structure of leftist trees of type $\lambda+1$.

This observation immediately implies the following result proved in [2].

**Lemma 1.** *Let*

$$H(z)=\sum_{n\geqslant 1} t(n)z^n \quad \textit{and} \quad T_\lambda(z)=\sum_{n\geqslant 1} t_\lambda(n)z^n$$

*be the generating function of the number $t(n)$ ($t_\lambda(n)$, $\lambda\in\mathrm{N}$) of all leftist trees (of type $\lambda$) with $n$ leaves. We have the following functional relation satisfied by $H(z)$*

$$H(z)=z+\tfrac{1}{2}H^2(z)+\tfrac{1}{2}\sum_{\lambda\geqslant 1} T_\lambda^2(z),$$

*where*

$$T_1(z)=z,$$

$$T_{\lambda+1}(z)=T_\lambda(z)[H(z)-\sum_{1\leqslant i<\lambda} T_i(z)], \quad \lambda\geqslant 1. \qquad \square$$

Since $t(n)=\sum_{\lambda\geqslant 1} t_\lambda(n)$, we also have

$$H(z)=\sum_{\lambda\geqslant 1} T_\lambda(z). \tag{3}$$

Starting with these relations, it can be verified (see [2]) that $H(z)$ has an algebraic singularity at $\eta=0.363\,704\,091\,587\ldots$. Furthermore, $H(z)$ is regular for $|z|\leqslant\eta$ except $z=\eta$; it has an expansion around $\eta$ of the form

$$H(z)=a+\sum_{k\geqslant 1} b_k(\eta-z)^{k/2}, \tag{4}$$

where

$$a=H(\eta)=0.823\,299\,737\,552\ldots \tag{5}$$

and

$$b_1=-2\sqrt{\pi/\eta}\,\alpha=-1.471\,640\,496\,987\ldots. \tag{6}$$

Let us conclude this section by the derivation of an asymptotic equivalent to the number $t_\lambda(n)$ of leftist trees of type $\lambda$ with $n$ leaves. For this purpose, we introduce the complex valued function $f_\lambda(z,y)$ defined by

$$f_1(z,y)=z,$$

$$f_{\lambda+1}(z,y)=f_\lambda(z,y)\Big[y-\sum_{1\leqslant i<\lambda} f_i(z,y)\Big],\quad \lambda\geqslant 1. \tag{7}$$

The result established in Lemma 1 immediately implies that the generating function $T_\lambda(z)$ of the numbers $t_\lambda(n)$ satisfies the relation $f_\lambda(z,H(z))=T_\lambda(z)$. Thus, using (4) and (7), an induction on $\lambda$ shows that

$$T_\lambda(z)=\alpha_\lambda+\sum_{k\geqslant 1}\tau_{k,\lambda}(\eta-z)^{k/2}, \tag{8}$$

where $\alpha_\lambda=f_\lambda(\eta,a)$ and $\tau_{1,\lambda}=b_1h_\lambda(\eta,a)$; here, $h_\lambda(z,y)$ stands for the partial derivative of $f_\lambda(z,y)$ with respect to $y$, that is

$$h_1(z,y)=0,$$

$$h_{\lambda+1}(z,y)=h_\lambda(z,y)\Big[y-\sum_{1\leqslant i<\lambda} f_i(z,y)\Big]+f_\lambda(z,y)\Big[1-\sum_{1\leqslant i<\lambda} h_i(z,y)\Big],$$

$$\lambda\geqslant 1. \tag{9}$$

By (3), formula (4) and (8) imply $\sum_{i\geqslant 1} f_i(\eta,a)=a$ and $\sum_{i\geqslant 1} h_i(\eta,a)=1$. Thus, using (7) and (9), it is easily verified that the sequences $f_\lambda(\eta,a)$ and $h_\lambda(\eta,a)$ tend exponentially to zero. Now, applying the Darboux-Pólya theorem ([1]) to the expansion (8), we immediately find the following result.

**Theorem 1.** *The number $t_\lambda(n)$ of leftist trees of type $\lambda\in\mathbf{N}$ with $n$ leaves satisfies*

$$t_\lambda(n)\sim\alpha h_\lambda(\eta,a)\eta^{-n}n^{-3/2},$$

*where $\alpha, \eta, a$ and $h_\lambda(\eta, a)$ are the constants defined in* (1), (5) *and by* (9). □

By an application of (1), we further obtain the following corollary.

**Corollary 1.** *The probability $p_\lambda(n)$ that a random leftist tree with $n$ leaves is of type $\lambda$ is asymptotically given by $p_\lambda(n) \sim h_\lambda(\eta, a)$.* □

For large $n$, the first few values of the probability $p_\lambda(n)$ and of the cumulative distribution function $V_\lambda(n) := \sum_{1 \leqslant k \leqslant \lambda} p_k(n)$ are summarized in Table 2(a). Comparing these values with the corresponding values of random extended binary trees given in Table 2(b) (set $r := \lambda - 1$ in Theorem 2 in [5]), we see that the cumulative distribution function $V_\lambda(n)$ of random leftist trees with $n$ leaves ascends steeper than that of random extended binary trees.

Table 2
Asymptotic values of the probability $p_\lambda(n)$ and of the cumulative distribution function $V_\lambda(n)$.

(a) Leftist Trees

| $\lambda$ | $p_\lambda(n) \sim h_\lambda(\eta, a)$ | $V_\lambda(n)$ |
|---|---|---|
| 1 | 0.000 000 | 0.000 000 |
| 2 | 0.363 704 | 0.363 704 |
| 3 | 0.466 594 | 0.830 298 |
| 4 | 0.162 296 | 0.992 594 |
| 5 | 0.007 398 | 0.999 992 |
| 6 | 0.000 007 | 0.999 999 |
| $\geqslant 7$ | 0.000 000 | 1.000 000 |

(b) Extended Binary Trees

| $\lambda$ | $p_\lambda(n) \sim 2^{-\lambda}(\lambda - 1)$ | $V_\lambda(n)$ |
|---|---|---|
| 1 | 0.000 000 | 0.000 000 |
| 2 | 0.250 000 | 0.250 000 |
| 3 | 0.250 000 | 0.500 000 |
| 4 | 0.187 500 | 0.687 500 |
| 5 | 0.125 000 | 0.812 500 |
| 6 | 0.078 125 | 0.890 625 |
| 7 | 0.046 875 | 0.937 500 |

For example, more than 99% of all leftist trees with $n$ leaves have a left branch length less than or equal to 3 (i.e. are of type $\leqslant 4$) while only 68.75% of all extended binary trees with $n$ leaves reach this value.

## 2. The average weight

In this section we shall derive the exact asymptotic behaviour of the average weight $\underline{w}(n)$ of a random leftist tree with $n$ leaves provided that the weight functions $\Phi_i(n)$, $i \in \{1, 2\}$, are arbitrary polynomials in the variable $n$ and that the weight function $g(\lambda, n)$ is an arbitrary polynomial in the

variable $n$ with coefficients depending on $\lambda$. For this purpose, we first introduce the generating functions

$$G_\lambda(z)=\sum_{n\geq 1} g(\lambda,n)\,z^n \tag{10}$$

and

$$F_i(z)=\sum_{n\geq 1} \Phi_i(n)\,z^n, \quad i\in\{1,2\}. \tag{11}$$

Let now $t(n,w)$ ($t_\lambda(n,w)$) be the number of all leftist trees (of type $\lambda$) with $n$ leaves and weight $w$. Obviously,

$$t(n,w)=\sum_{\lambda\geq 1} t_\lambda(n,w), \quad t_\lambda(n)=\sum_{w\geq 0} t_\lambda(n,w), \quad t(n)=\sum_{w\geq 0} t(n,w). \tag{12}$$

The average weight $\underline{w}(n)$ ($\underline{w}_\lambda(n)$) of a random leftist tree (of type $\lambda$) with $n$ leaves is given by the expected value

$$\underline{w}(n)=\frac{1}{t(n)}\sum_{w\geq 0} wt(n,w) \quad \left(\underline{w}_\lambda(n)=\frac{1}{t_\lambda(n)}\sum_{w\geq 0} wt_\lambda(n,w)\right). \tag{13}$$

Thus, $\underline{w}(n)=\langle z^n;\, Y(z)\rangle/t(n)$[4] and $\underline{w}_\lambda(n)=\langle z^n;\, Y_\lambda(z)\rangle/t_\lambda(n)$, where

$$Y(z)=\sum_{n\geq 1} t(n)\,\underline{w}(n)\,z^n \quad \text{and} \quad Y_\lambda(z)=\sum_{n\geq 1} t_\lambda(n)\,\underline{w}_\lambda(n)\,z^n. \tag{14}$$

Note that by (12) and the definition of the weight

$$Y(z)=\sum_{\lambda\geq 1} Y_\lambda(z) \quad \text{and} \quad Y_1(z)=zg(1,1). \tag{15}$$

Now by means of these notations, we are ready to establish the following rather general result.

**Theorem 2.** *The average weight $\underline{w}(n)$ of a random leftist tree with $n$ leaves is given by $\underline{w}(n)=\langle z^n;\, Y(z)\rangle/t(n)$, where $Y(z)=\sum_{\lambda\geq 1} Y_\lambda(z)$ and $Y_\lambda(z)$ satisfies the recurrence*

[4]) The abbreviation $\langle z^n;\, f(z)\rangle$ denotes the coefficient of $z^n$ in the expansion of $f(z)$ at $z=0$.

$$Y_1(z)=zg(1,1),$$

$$Y_{\lambda+1}(z)=[T_{\lambda+1}(z)\odot G_{\lambda+1}(z)]+(c_1Y_\lambda(z)+[T_\lambda(z)\odot F_1(z)])(H(z)-\sum_{1\leqslant r<\lambda}T_r(z))$$

$$+T_\lambda(z)\{c_2(Y(z)-\sum_{1\leqslant r<\lambda}Y_r(z))+[(H(z)-\sum_{1\leqslant r<\lambda}T_r(z))\odot F_2(z)]\},$$

$\lambda\geqslant 1.$

*Here, $h_1(z)\odot h_2(z)$ denotes the Hadamard product of the two power series $h_i(z)=\sum_{n\geqslant 0}h_{i,n}z^n$, $i\in\{1,2\}$, defined by $h_1(z)\odot h_2(z)=\sum_{n\geqslant 0}h_{1,n}h_{2,n}z^n$. The functions $T_\lambda(z)$ and $H(z)$ are defined in* (8) *and by Lemma* 1.

**Proof.** Let $\lambda\geqslant 1$. The leftist tree of type $(\lambda+1)$ drawn in Figure 2 has $n$ leaves and weight $w-g(\lambda+1,n)$ if and only if the subtree $T_1$ ($T_2$) is a leftist tree of type $\lambda$ (of type $r\geqslant\lambda$) with $n_1$ ($n_2$) leaves and weights $w_1$ ($w_2$), where $n=n_1+n_2$ and $w-g(\lambda+1,n)=c_1w_1+c_2w_2+\Phi_1(n_1)+\Phi_2(n_2)$. Translating this fact into terms of the generating functions

$$E_i(z,y)=\sum_{n\geqslant 1}\sum_{w\geqslant 0}t(n,w)z^n y^{c_iw+\Phi_i(n)},\quad i\in\{1,2\},$$

$$E_i^{[\lambda]}(z,y)=\sum_{n\geqslant 1}\sum_{w\geqslant 0}t_\lambda(n,w)z^n y^{c_iw+\Phi_i(n)},\quad i\in\{1,2\},$$

and

$$R_\lambda(z,y)=\sum_{n\geqslant 1}\sum_{w\geqslant 0}t_\lambda(n,w)z^n y^{w-g(\lambda,n)},$$

we immediately find the relations

$$R_{\lambda+1}(z,y)=E_1^{[\lambda]}(z,y)\sum_{r\geqslant\lambda}E_2^{[r]}(z,y)$$

$$=E_1^{[\lambda]}(z,y)[E_2(z,y)-\sum_{1\leqslant r<\lambda}E_2^{[r]}(z,y)]. \tag{16}$$

Note that $E_i(z,1)=H(z)$ and $E_i^{[\lambda]}(z,1)=R_\lambda(z,1)=T_\lambda(z)$, $i\in\{1,2\}$. Using this relation together with (12), (13) and the above definitions of $G_\lambda(z)$, $F_i(z)$, $Y(z)$, $Y_\lambda(z)$, $E_i(z,y)$, $E_i^{[\lambda]}(z,y)$ and $R_\lambda(z,y)$, we find the recursive definition of $Y_\lambda(x)$ established in our theorem by taking the partial derivative with

respect to $y$ on both sides of the equation (16) and by setting then $y:=1$. □

The result presented in the preceding theorem is rather technical and leads to an intricate definition of $Y(z)$ by means of $Y_\lambda(z)$. On the other hand, we see that the recurrence for $Y_\lambda(z)$ is a linear inhomogeneous difference equation with full history; furthermore, each $Y_\lambda(z)$ linearly depends on the function $Y(z)$. Indeed, an induction on $\lambda$ shows that

$$Y_\lambda(z)=A_\lambda(z)+Y(z)\,B_\lambda(z), \tag{17}$$

where

$$A_1(z)=zg(1,1),$$

$$\begin{aligned} A_{\lambda+1}(z)=&[T_{\lambda+1}(z)\odot G_{\lambda+1}(z)]\\ &+(c_1A_\lambda(z)+[T_\lambda(z)\odot F_1(z)])(H(z)-\sum_{1\leqslant r<\lambda}T_r(z))\\ &-T_\lambda(z)\{c_2\sum_{1\leqslant r<\lambda}A_r(z)-[H(z)-\sum_{1\leqslant r<\lambda}T_r(z)]\odot F_2(z)\},\ \lambda\geqslant 1, \end{aligned} \tag{18}$$

and

$$B_1(z)=0,$$

$$B_{\lambda+1}(z)=c_1B_\lambda(z)(H(z)-\sum_{1\leqslant r<\lambda}T_r(z))+c_2T_\lambda(z)(1-\sum_{1\leqslant r<\lambda}B_r(z)),\ \lambda\geqslant 1. \tag{19}$$

Henceforth, we restrict our further considerations to the case that the functions $Y_\lambda(z)$ and $Y(z)$ are regular for $|z|\leqslant\eta$ except $z=\eta$. Furthermore, we assume that the functions $\Phi_i(n)$, $i\in\{1,2\}$, are arbitrary polynomials in $n$ and that the function $g(\lambda,n)$ is an arbitrary polynomial in $n$ with coefficients depending on $\lambda$. We choose the representations

$$\Phi_i(n)=\sum_{0\leqslant p\leqslant d_i}\varphi_{i,p}n^p \quad\text{with}\quad \varphi_{i,d_i}\not\equiv 0,\ i\in\{1,2\}, \tag{20a}$$

and

$$g(\lambda,n)=\sum_{0\leqslant p\leqslant \rho}\psi_p(\lambda)n^p \quad\text{with}\quad \psi_p(\lambda)\not\equiv 0. \tag{20b}$$

With these assumptions, we are able to cancel the cumbersome

Hadamard products in the above definition of $A_\lambda(z)$. For example, the Hadamard product $T_\lambda(z) \odot G_\lambda(z)$ can be transformed as follows:

$$\begin{aligned} T_\lambda(z) \odot G_\lambda(z) &= \sum_{n \geqslant 1} t_\lambda(n) g(\lambda, n) z^n \\ &= \sum_{0 \leqslant p \leqslant \rho} \psi_p(\lambda) \sum_{n \geqslant 1} t_\lambda(n) n^p z^n \\ &= \sum_{0 \leqslant p \leqslant \rho} \psi_p(\lambda) \sum_{n \geqslant 1} t_\lambda(n) z^n \sum_{0 \leqslant i \leqslant p} i! \binom{n}{i} s_p^{(i)} \\ &= \sum_{0 \leqslant p \leqslant \rho} \psi_p(\lambda) \sum_{0 \leqslant i \leqslant p} s_p^{(i)} z^i \frac{d^i}{dz^i} T_\lambda(z). \end{aligned}$$

Here, we have used the well-known relation $x^p = \sum_{0 \leqslant i \leqslant p} i! \binom{x}{i} s_p^{(i)}$, where $s_p^{(i)}$ is a Stirling number of the second kind.

Doing this procedure for all Hadamard products appearing in $A_\lambda(z)$, we find by a lengthy computation

$$A_1(z) = z g(1, 1),$$

and for $\lambda \geqslant 1$

$$\begin{aligned} A_{\lambda+1}(z) = &\sum_{0 \leqslant p \leqslant \rho} \psi_p(\lambda+1) \sum_{0 \leqslant i \leqslant p} s_p^{(i)} z^i \frac{d^i}{dz^i} T_{\lambda+1}(z) \\ &+ (c_1 A_\lambda(z) + \sum_{0 \leqslant p \leqslant d_1} \varphi_{1,p} \sum_{0 \leqslant i \leqslant p} s_p^{(i)} z^i \frac{d^i}{dz^i} T_\lambda(z)) (H(z) - \sum_{1 \leqslant r < \lambda} T_r(z)) \\ &- T_\lambda(z) [c_2 \sum_{1 \leqslant r < \lambda} A_r(z) - \sum_{0 \leqslant p \leqslant d_2} \varphi_{2,p} \sum_{0 \leqslant i \leqslant p} s_p^{(i)} z^i \frac{d^i}{dz^i} (H(z) - \sum_{1 \leqslant r < \lambda} T_r(z))]. \end{aligned}$$

Our next aim is to compute an evaluation of $A_\lambda(z)$ and $B_\lambda(z)$ in the neighbourhood of the algebraic singularity $\eta$. First, let us consider $A_\lambda(z)$. Using the expansions (4), (8) and the above recursive definition of $A_\lambda(z)$, an induction on $\lambda$ shows that

- If $MAX\{\rho, d_1, d_2\} = 0$, then $A_\lambda(z) = \mu_\lambda + \sum_{k \geqslant 1} \vartheta_{k,\lambda} (\eta - z)^{k/2}$, where

$$\mu_1 = \eta g(1,1),$$

$$\mu_{i+1} = [(\psi_0(i+1) + \varphi_{1,0} + \varphi_{2,0}) f_i(\eta, a) + c_1 \mu_i]\,[a - \sum_{1 \leqslant r < i} f_r(\eta, a)]$$

$$- c_2 f_i(\eta, a) \sum_{1 \leqslant r < i} \mu_r, \quad i \geqslant 1, \tag{21}$$

and

$$\vartheta_{1,1} = 0,$$

$$\vartheta_{1,i+1} = [c_1 \vartheta_{1,i} + (\varphi_{1,0} + \varphi_{2,0}) b_1 h_i(\eta, a)]\,[a - \sum_{1 \leqslant r < i} f_r(\eta, a)]$$

$$+ b_1 [c_1 \mu_i + (\varphi_{1,0} + \varphi_{2,0}) f_i(\eta, a)]\,[1 - \sum_{1 \leqslant r < i} h_r(\eta, a)]$$

$$+ b_1 \psi_0(i+1) h_{i+1}(\eta, a) - c_2 f_i(\eta, a) \sum_{1 \leqslant r < i} \vartheta_{1,r}$$

$$- c_2 b_1 h_i(\eta, a) \sum_{1 \leqslant r < i} \mu_r, \quad i \geqslant 1. \tag{22}$$

- If $MAX\{\rho, d_1, d_2\} = m \geqslant 1$, then $A_\lambda(z) = \sum_{k \geqslant 1} \xi_{k,\lambda} (\eta - z)^{-(2m-k)/2}$, where

$$\xi_{1,1} = 0,$$

$$\xi_{1,i+1} = -\frac{(2m-2)!}{(m-1)!} 2^{-2m+1} b_1 \eta^m [\psi_m(i+1) h_{i+1}(\eta, a)$$

$$+ \varphi_{1,m} h_i(\eta, a)\,(a - \sum_{1 \leqslant r < i} f_r(\eta, a))$$

$$+ \varphi_{2,m} f_i(\eta, a)(1 - \sum_{1 \leqslant r < i} h_r(\eta, a))]$$

$$+ c_1 \xi_{1,i} [a - \sum_{1 \leqslant r < i} f_r(\eta, a)] - c_2 f_i(\eta, a) \sum_{1 \leqslant r < i} \xi_{1,r}, \quad i \geqslant 1. \tag{23}$$

By (4), (8) and (19), another induction on $\lambda$ shows that $B_\lambda(z) = \gamma_\lambda + \sum_{k \geqslant 1} \beta_{k,\lambda} (\eta - z)^{k/2}$ where

$$\gamma_1=0,$$

$$\gamma_{i+1}=c_1\gamma_i[a-\sum_{1\leqslant r<i} f_r(\eta,a)]+c_2 f_i(\eta,a)[1-\sum_{1\leqslant r<i}\gamma_r],\quad i\geqslant 1, \tag{24}$$

and

$$\beta_{1,1}=0,$$

$$\begin{aligned}\beta_{1,i+1}=c_1b_1\gamma_i[1-\sum_{1\leqslant r<i} h_r(\eta,a)]+c_1\beta_{1,i}[a-\sum_{1\leqslant r<i} f_r(\eta,a)]\\ +c_2b_1h_i(\eta,a)[1-\sum_{1\leqslant r<i}\gamma_r]-c_2f_i(\eta,a)\sum_{1\leqslant r<i}\beta_{1,r},\ i\geqslant 1.\end{aligned} \tag{25}$$

Note that the sequences defined in (21)-(25) and the corresponding sums over $i$ must converge because we have assumed that $Y(z)$ and $Y_\lambda(z)$ are regular when $|z|\leqslant\eta$, except $z=\eta$. Henceforth, the values of the sums $\sum_{\lambda\geqslant 1} o_\lambda$, $o\in\{\mu,\vartheta_1,\xi_1,\gamma,\beta_1\}$ are denoted by $\mu,\vartheta,\xi,\gamma$ and $\beta$, respectively.

Since by (15) and by (17)

$$Y(z)=\frac{\sum_{\lambda\geqslant 1} A_\lambda(z)}{1-\sum_{\lambda\geqslant 1} B_\lambda(z)}, \tag{26}$$

we obtain the evaluation of $Y(z)$ around the algebraic singularity $\eta$. Applying now the Darboux-Pólya theorem ([1]) to this evaluation, we find the following rather general result by means of (1) and (6).

**Theorem 3.** *Let the weight $w(T)$ of a leftist tree $T$ be defined as in Section 1 and assume that the functions $\Phi_i(n)$, $i\in\{1,2\}$, and $g(\lambda,n)$ satisfy the restrictions established in* (20a) *and* (20b). *If the sums over $i$ of the sequences defined in* (21)-(25) *converge, then the average weight $\underline{w}(n)$ of a random leftist tree with $n$ leaves is asymptotically given by:*

(a) *If $MAX\{\rho,d_1,d_2\}=0$ and $\gamma\neq 1$, then*

$$\underline{w}(n)=\frac{(1-\gamma)\vartheta+\mu\beta}{(1-\gamma)^2 b_1}+O\left(\frac{1}{n}\right),$$

*where the terms $\mu_i,\vartheta_{1,i},\gamma_i$ and $\beta_{1,i}$ appearing in the sums $\mu,\vartheta,\gamma$ and $\beta$ are given by* (21), (22), (24) *and* (25), *respectively.*

(b) *If* $MAX\{\rho, d_1, d_2\}=0$ *and* $\gamma=1$, *then*

$$\underline{w}(n)=\frac{2\mu}{\eta\beta b_1}n+O(1),$$

*where the terms* $\mu_i, \gamma_i$ *and* $\beta_i$ *appearing in the sums* $\mu, \gamma$ *and* $\beta$ *are given by* (21), (24) *and* (25), *respectively.*

(c) *If* $MAX\{\rho, d_1, d_2\}=m\geqslant 1$ *and* $\gamma\neq 1$, *then*

$$\underline{w}(n)=-2^{2m-1}\eta^{-m}\frac{(m-1)!\xi}{(2m-2)!\,(1-\gamma)\,b_1}n^m+O(n^\alpha),$$

$$\alpha=m-\tfrac{1}{2}(1+\delta_{m,1}),$$

*where the terms* $\xi_{1,i}$ *and* $\gamma_i$ *appearing in the sums* $\xi$ *and* $\gamma$ *are given by* (23) *and* (24), *respectively.*

(d) *If* $MAX\{\rho, d_1, d_2\}=m\geqslant 1$ *and* $\gamma=1$, *then*

$$\underline{w}(n)=\sqrt{\frac{\pi}{\eta}}\eta^{-m}\frac{2\xi}{(m-1)!\,\beta\, b_1}n^{m+1/2}+O(n^m),$$

*where the terms* $\xi_{1,i}$, $\gamma_i$ *and* $\beta_{1,i}$ *appearing in the sums* $\xi$, $\gamma$ *and* $\beta$ *are given by* (23), (24) *and* (25), *respectively.* □

## 3. Applications

In this section, we shall apply the general result presented in the preceding theorem to the particular weights defined in Table 1. The first few values of these expected values are summarized in the Appendix.

### 3.1. *The average left branch length*

Using Table 1, (20a) and (20b), we find the parameters $(\rho, d_1, d_2)=(0,0,0)$, $(c_1, c_2)=(1,0)$, $\varphi_{i,0}=0$ for $i\in\{1,2\}$, and $\psi_0(\lambda)=1-\delta_{\lambda,1}$. Since the recurrence for $\gamma_i$ established in (24) immediately implies $\gamma_i=0$ for all $i\geqslant 1$, we have to apply part (a) of Theorem 3. Thus

$$\underline{LBL}(n) \sim \vartheta / b_1,$$

because the recurrence for $\beta_{1,i}$ also leads to $\beta_{1,i}=0$ for all $i \geqslant 1$. Using (7), the recurrence for $\mu_i$ given in (21) implies $\mu_i=(i-1)f_i(\eta, a)$ for all $i \geqslant 1$. Inserting this expression for $\mu_i$ into the recurrence for $\vartheta_{1,i}$ presented in (22), we further find by use of (9) that $\vartheta_{1,i}=b_1(i-1)\,h_i(\eta, a)$ for all $i \geqslant 1$. Since $\vartheta=-2.668\,688\,253\,013...$, we have just proved the following result.

**Theorem 4.** *The average left branch length $\underline{LBL}(n)$ of a random leftist with $n$ leaves is asymptotically given by*

$$\underline{LBL}(n) \sim 1.813\,410\,448\,051... \; (=const.). \qquad \square$$

Obviously, this result can also be derived by a direct application of (8): Introducing the function

$$M_s(z)=\sum_{\lambda \geqslant 1} (\lambda-1)^s\, T_\lambda(z), \quad s \in \mathrm{N} \text{ fixed},$$

then $\underline{LBL}(n)=\langle z^n; M_1(n)\rangle / t(n)$ and we immediately obtain the value of $\underline{LBL}(n)$ by (1) and the Darboux-Pólya theorem. In the same way, the second moment $m_2(n):=\langle z^n; M_2(z)\rangle / t(n)$ about the origin can be computed. For the variance, we find $\sigma^2(n)=m_2(n)-m_1^2(n) \sim 0.520\,843\,544...$, which is very small. Note that the corresponding values for extended binary trees with $n$ leaves are $\underline{LBL}(n) \sim 3$ and $\sigma^2(n) \sim 4$ (see [5]).

### 3.2. *The average right branch length*

Using again Table 1, (20a) and (20b), we obtain $(\rho, d_1, d_2)=(0,0,0)$, $(c_1, c_2)=(0,1)$, $\varphi_{i,0}=0$ for $i \in \{1,2\}$ and $\psi_0(\lambda)=1-\delta_{\lambda,1}$. Here, $\gamma_i$ satisfies the recurrence

$$\gamma_1=0, \quad \gamma_{i+1}=f_i(\eta, a)\,[1-\sum_{1 \leqslant r < i} \gamma_r], \quad i \geqslant 1,$$

which implies $\gamma=0.758\,257\,313\,346... \neq 0$. Again, we have to apply part (a) of Theorem 3 and find

$$\underline{RBL}(n) \sim \frac{(1-\gamma)\vartheta+\mu\beta}{(1-\gamma)^2\, b_1},$$

where by (22)

$$\vartheta_{1,1}=0,\quad \vartheta_{1,i+1}=b_1h_{i+1}(\eta,a)-f_i(\eta,a)\sum_{1\leqslant r<i}\vartheta_{1,r}-b_1h_i(\eta,a)\sum_{1\leqslant r<i}\mu_r,$$

$$i\geqslant 1,$$

and by (21)

$$\mu_1=0,\quad \mu_{i+1}=f_i(\eta,a)\Big[a-\sum_{1\leqslant r<i}f_r(\eta,a)\Big]-f_i(\eta,a)\sum_{1\leqslant r<i}\mu_r,\quad i\geqslant 1,$$

and by (25)

$$\beta_{1,1}=0,\quad \beta_{1,i+1}=b_1h_i(\eta,a)\Big[1-\sum_{1\leqslant r<i}\gamma_r\Big]-f_i(\eta,a)\sum_{1\leqslant r<i}\beta_{1,r},\ i\geqslant 1.$$

Since $\vartheta=-1.055\,908\,236\,644...$, $\mu=0.408\,546\,134\,338...$, and $\beta=-1.043\,052\,105\,273...$, we have just derived the following result.

**Theorem 5.** *The average right branch length* $\underline{RBL}(n)$ *of a random leftist tree with* $n$ *leaves is asymptotically given by*

$$\underline{RBL}(n)\sim 7.922\,992\,562\,093...\ (=const.).$$

As expected, the average right branch length of a random leftist tree with $n$ leaves is essentially higher than the average left branch length. Since the family of extended binary trees with $n$ leaves is closed under the permutation of the left and right subtree of the root, the corresponding value for a random extended binary tree is $\underline{RBL}(n)=\underline{LBL}(n)\sim 3$.

### 3.3. *The average external path length*

Here, Table 1, (20a) and (20b) yield $(\rho,d_1,d_2)=(0,1,1)$, $(c_1,c_2)=(1,1)$, $\varphi_{i,0}=0$, $\varphi_{i,1}=1$ for $i\in\{1,2\}$ and $\psi_0(\lambda)=0$. Using (9), it is easily verified that the recurrence for $\gamma_i$ given in (24) implies $\gamma_i=h_i(\eta,a)$ for all $i\geqslant 1$. Since $\sum_{i\geqslant 1}h_i(\eta,a)=1$, we have to apply part (d) of Theorem 3 with $m=1$. We find

$$\underline{EPL}(n)\sim\sqrt{\frac{\pi}{\eta}}\,\frac{2\xi}{\eta\beta b_1}\,n^{3/2},$$

where by (9), (23) and (25)

$$\xi_{1,1}=0,\ \xi_{1,i+1}=-\tfrac{1}{2}b_1\eta\, h_{i+1}(\eta,a)+\xi_{1,i}[a-\sum_{1\leqslant r<i} f_r(\eta,a)]$$
$$-f_i(\eta,a)\sum_{1\leqslant r<i}\xi_{1,r}$$

and

$$\beta_{1,1}=0,$$
$$\beta_{1,i+1}=2b_1h_i(\eta,a)[1-\sum_{1\leqslant r<i} h_r(\eta,a)]+\beta_{1,i}[a-\sum_{1\leqslant r<i} f_r(\eta,a)]$$
$$-f_i(\eta,a)\sum_{1\leqslant r<i}\beta_{1,r}.$$

Since $\xi=0.321\,367\,083\,017...$ and $\beta=-2.195\,944\,277\,826...$ we obtain the following result.

**Theorem 6.** *The average external path length $\underline{EPL}(n)$ of a random leftist tree with $n$ leaves is asymptotically given by*

$$\underline{EPL}(n)\sim 1.607\,166\,963\,486...n^{3/2}. \qquad \square$$

Thus by (2a), the average internal path length $\underline{IPL}(n)$ of a random leftist tree with $n$ leaves is also given by Theorem 6. Note that the corresponding values for random extended binary trees with $n$ leaves are $\underline{EPL}(n)\sim\underline{IPL}(n)\sim\sqrt{\pi}\,n^{3/2}=1.772\,453\,851...\,n^{3/2}$ (see [6; p. 404,509]).

### 3.4. *The average external free path length*

In this case, Table 1, (20a) and (20b) lead to

$$\underline{EFPL}(n)=n\,\underline{EPL}(n)-\underline{w}(n),$$

where $\underline{w}(n)$ is the average weight induced by the parameters $g(\lambda,n)=0$, $(c_1,c_2)=(1,1)$ and $\Phi_i(n)=n^2$, $i\in\{1,2\}$, that is $(\rho,d_1,d_2)=(0,2,2)$, $\varphi_{i,0}=\varphi_{i,1}=0$, $\varphi_{i,2}=1$ for $i\in\{1,2\}$, and $\psi_0(\lambda)=0$. In the same way as in Section 3.3, we obtain $\gamma_i=h_i(\eta,a)$ for all $i\geqslant 1$, and part (d) of Theorem 3 with $m=2$ has to be applied. We find

$$\underline{w}(n)\sim\sqrt{\frac{\pi}{\eta}}\frac{2\xi}{\eta^2\beta b_1}n^{5/2}.$$

Comparing the recurrences for $\xi_{1,i}$ and $\beta_{1,i}$ given in (23) and (25) with the corresponding relations established in Section 3.3, we immediately find the relation $\underline{w}(n) \sim \frac{1}{2} n\, \underline{EPL}(n)$. Thus, we obtain the following result.

**Theorem 7.** *The average external free path length $\underline{EFPL}(n)$ of a random leftist tree with n nodes is asymptotically given by*

$$\underline{EFPL}(n) \sim 0.803\,583\,481\,743\ldots\, n^{5/2}. \qquad \square$$

Thus by (2b) and (2c), the average internal (internal-external) free path length of a random leftist tree with $n$ leaves is determined by $\underline{IFPL}(n) \sim \underline{EFPL}(n)$ and $\underline{IEFPL}(n) \sim 2\underline{EFPL}(n)$.

Note that the corresponding values for random extended binary trees with $n$ leaves are $\underline{IFPL}(n) \sim \underline{EFPL}(n) \sim \frac{1}{2}\sqrt{\pi}\, n^{5/2} = 0.886\,226\,925\ldots\, n^{5/2}$ and $\underline{IEFPL}(n) \sim 2\underline{EFPL}(n)$ (see [4]).

### 3.5. *The average left path length*

Using Table 1, (20a) and (20b), we find the parameters $(\rho, d_1, d_2) = (0,0,0)$, $(c_1, c_2) = (1,1)$, $\varphi_{i,0} = 0$ for $i \in \{1,2\}$, and $\psi_0(\lambda) = \lambda - 1$. Thus by (9), the recurrence for $\gamma_i$ established in (24) implies $\gamma_i = h_i(\eta, a)$ for all $i \geqslant 1$ and we have to apply part (b) of Theorem 3 because $\sum_{\lambda \geqslant 1} h_\lambda(\eta, a) = 1$. We obtain

$$\underline{LPL}(n) \sim \frac{2\mu}{\eta \beta b_1} n.$$

The recurrence for $\mu_i$ is

$$\mu_1 = 0, \quad \mu_{i+1} = [i f_i(\eta, a) + \mu_i]\,[a - \sum_{1 \leqslant r < i} f_r(\eta, a)] - f_i(\eta, a) \sum_{1 \leqslant r < i} \mu_r,$$

and the recurrence for $\beta_{1,i}$ is identical to that given in Section 3.3. Since $\mu = 0.791\,270\,556\,391\ldots$ and $\beta = -2.195\,944\,277\,826\ldots$, we obtain the following result.

**Theorem 8.** *The average left path length $\underline{LPL}(n)$ of a random leftist tree with n leaves is asymptotically given by*

$$\underline{LPL}(n) \sim 1.346\,429\,965\,522\ldots n. \qquad \square$$

Note that the corresponding value for random extended binary trees with $n$ leaves is $\underline{LPL}(n) \sim 2n$ (see [3]).

Unfortunately, the right length $\underline{RPL}(n)$ of a leftist tree cannot be defined by a weight $w(T)$ introduced in Section 1. But on the other hand, the asymptotic behaviour of $\underline{RPL}(n)$ can easily be determined by the general approach given in Section 2. This will be done in Section 4.

### 3.6. *The average number of root-free paths*

Here, Table 1, (20a) and (20b) lead to $(\rho, d_1, d_2) = (0, 2, 2)$, $(c_1, c_2) = (0, 0)$, $\varphi_{i,0} = 0$, $\varphi_{i,1} = -0.5$, $\varphi_{i,2} = 0.5$ for all $i \in \{1, 2\}$, and $\psi_0(\lambda) = 0$. The recurrence for $\gamma_i$ given in (24) implies $\gamma_i = 0$ for all $i \geqslant 1$, and we have to apply part (c) of Theorem 3 with $m = 2$. We find

$$\underline{EP}_r(n) \sim -\frac{4\xi}{\eta^2 b_1} n^2.$$

Using (9), the recurrence for $\xi_{1,i}$ established in (23) implies $\xi_{1,i} = -\frac{1}{8} b_1 \eta^2 h_i(\eta, a)$. Since $\sum_{\lambda \geqslant 1} h_\lambda(\eta, a) = 1$, we obtain the following result.

**Theorem 9.** *The average number $\underline{EP}_r(n)$ of root-free paths between leaves in a random leftist tree with n leaves is asymptotically given by*

$$\underline{EP}_r(n) \sim \tfrac{1}{2} n^2. \qquad \square$$

Thus by (2d) and (2e), we also have $\underline{IP}_r(n) \sim n^2/2$ ($\underline{IEP}_r(n) \sim n^2$) for the average number of internal (internal-external) root-free paths of a random leftist tree with $n$ leaves. Note that the same results hold for extended binary trees with $n$ leaves (see [4]).

### 3.7. *The average number of leaves in the subtrees of the root*

Let us compute the average number of leaves in the right subtree of the root. Using Table 1, (20a) and (20b), we obtain $(\rho, d_1, d_2) = (0, 0, 1)$, $(c_1, c_2) = (0, 0)$, $\varphi_{1,0} = \varphi_{1,1} = \varphi_{2,0} = 0$, $\varphi_{2,1} = 1$ and $\psi_0(\lambda) = 0$. Here, the

recurrence for $\gamma_i$ established in (24) implies $\gamma_i=0$ for all $i\geqslant 1$ and we have to apply part (c) of Theorem 3 with $m=1$. We obtain

$$\underline{L}_r(n)\sim -\frac{2\xi}{\eta\, b_1}n,$$

where $\xi_{1,i}$ is defined by (23), that is,

$$\xi_{1,1}=0,\quad \xi_{1,i+1}=-\tfrac{1}{2}b_1\,\eta f_i(\eta,a)\,[1-\sum_{1\leqslant r<i} h_r(\eta,a)].$$

Since $\xi=0.201\,907\,286\,298...$, we find the following result.

**Theorem 10.** *The average number $\underline{L}_r(n)$ of leaves appearing in the right subtree of the root of a random leftist tree with n leaves is asymptotically given by*

$$\underline{L}_r(n)\sim 0.754\,452\,792\,364...\,n. \qquad \square$$

Thus, about 24.55% of all leaves in a random leftist tree with $n$ leaves appear in the left subtree of the root. Note that the corresponding values for random extended binary trees with $n$ leaves are $\underline{L}_r(n)=\underline{L}_1(n)=n/2$.

## 4. The average right path length

Using the notation presented in Table 1, the right path length of a leftist tree $T$ is defined by

$$RPL(T)=\sum_{v\in I} d(v,b_v).$$

As mentioned at the end of Section 3.5, the parameter $RPL(T)$ does not satisfy the recursive definition of a weight $w(T)$ given in Section 1; but, it can easily be defined by means of the right branch length $RBL(T)$ as follows

$$RPL(T)=RPL(T_1)+RPL(T_2)+RBL(T). \tag{26}$$

Here, $T_1$ ($T_2$) denotes the left (right) subtree of the root of $T$.

For the sake of completeness, we will summarize the main steps appearing in the computation of the average right path length $\underline{RPL}(n)$ of a random

leftist tree with $n$ leaves. For this purpose, let $t(n,r,s)$ $(t_\lambda(n,r,s))$ be the number of all leftist trees $T$ (of type $\lambda$) with $n$ leaves , $RPL(T)=r$ and $RBL(T)=s$, and let

$$D(z,u,t)=\sum_{n\geqslant 1}\sum_{r\geqslant 0}\sum_{s\geqslant 0} t(n,r,s)\,z^n u^r t^s$$

$$(D_\lambda(z,u,t)=\sum_{n\geqslant 1}\sum_{r\geqslant 0}\sum_{s\geqslant 0} t_\lambda(n,r,s)z^n u^r t^s)$$

be the corresponding generation function. An inspection of Table 1 shows that $RBL(T)=RBL(T_2)+1$. Translating this relation together with (26) into terms of the generating function $D(z,u,t)$ $(D_\lambda(z,u,t))$, we immediately find

$$\begin{aligned} D_1(z,u,t)&=z,\\ D_{\lambda+1}(z,u,t)&=utD_\lambda(z,u,1)\sum_{r\geqslant\lambda}D_r(z,u,ut)\\ &=utD_\lambda(z,u,1)\,[D(z,u,ut)-\sum_{1\leqslant r<\lambda}D_r(z,u,ut)],\ \lambda\geqslant 1, \end{aligned} \tag{27}$$

because

$$D(z,u,t)=\sum_{\lambda\geqslant 1}D_r(z,u,t).$$

Obviously,

$$\underline{RPL}(n)=\frac{1}{t(n)}\sum_{n\geqslant 1}r\sum_{s\geqslant 0}t(n,r,s)=\frac{1}{t(n)}<z^n;\frac{\partial}{\partial u}D(z,u,t)\,|_{(u,t)=(1,1)}>.$$

Setting $\Delta(z):=\frac{\partial}{\partial u}D(z,u,t)\,|_{(u,t)=(1,1)}$ and $\Delta_\lambda(z):=\frac{\partial}{\partial u}D_\lambda(z,u,t)\,|_{(u,t)=(1,1)}$, formula (27) implies

$$\begin{aligned} \Delta_1(z)&=0,\\ \Delta_{i+1}(z)&=T_i(z)\,[H(z)-\sum_{1\leqslant r<i}T_r(z)]+\Delta_i(z)[H(z)-\sum_{1\leqslant r<i}T_r(z)]\\ &\quad+T_i(z)[\Delta(z)+\frac{\partial}{\partial t}D(z,1,1)-\sum_{1\leqslant r<i}(\Delta_r(z)+\frac{\partial}{\partial t}D_r(z,1,1))],\\ i&\geqslant 1. \end{aligned} \tag{28}$$

Here, we have used the obvious relations $D_i(z,1,1)=T_i(z)$ and $D(z,1,1)=H(z)$. Note that $\frac{\partial}{\partial t}D(z,1,1)$ $(\frac{\partial}{\partial t}D_\lambda(z,1,1))$ is just the generating function of the sum of the right branch lengths of all leftist trees (of type $\lambda$) with $n$ leaves, that is

$$\frac{\partial}{\partial t}D(z,1,1)=Y(z) \quad \text{and} \quad \frac{\partial}{\partial t}D_\lambda(z,1,1)=Y_\lambda(z),$$

where $Y(z)$ and $Y_\lambda(z)$ are the functions defined in (14) with $\underline{w}(n)=\underline{RBL}(n)$ and $\underline{w}_\lambda(n)=\underline{RBL}_\lambda(n)$. Now, Table 1, the results of Section 3.2 and the formulas (21), (22), (24) and (26) yield the expansion of $Y(z)$ and $Y_\lambda(z)$ around the algebraic singularity $\eta$. We find

$$Y(z)=\frac{\mu}{1-\gamma}+(\eta-z)^{1/2}\frac{(1-\gamma)\vartheta+\mu\beta}{(1-\gamma)^2}+\text{terms}\,(\eta-z)^{k/2}, \quad k\geqslant 2, \tag{29}$$

and

$$\begin{aligned} Y_\lambda(z)=&\frac{(1-\gamma)\mu_\lambda+\mu\gamma_\lambda}{1-\gamma}\\ &+(\eta-z)^{1/2}\frac{\vartheta_{1,\lambda}(1-\gamma)^2+\mu\beta_{1,\lambda}(1-\gamma)+\gamma_\lambda[(1-\gamma)\vartheta+\mu\beta]}{(1-\gamma)^2}\\ &+\text{terms}\,(\eta-z)^{k/2}, \quad k\geqslant 2, \end{aligned} \tag{30}$$

where $\mu_i$, $\vartheta_{1,i}$ and $\beta_{1,i}$ are recursively defined in Section 3.2, and $\mu, \vartheta, \gamma$ and $\beta$ are the values of the corresponding sums according to the convention after formula (25). Now, we can proceed in a similar way as we have done following Theorem 2. The recurrence (28) is a linear inhomogeneous difference equation with full history and each $\Delta_\lambda(z)$ linearly depends on $\Delta(z)$, that is $\Delta_\lambda(z)$ can be written in the form

$$\Delta_\lambda(z)=K_\lambda(z)+\Delta(z)\,B_\lambda(z), \tag{31}$$

where $B_\lambda(z)$ is defined by the recurrence (19) with $(c_1,c_2)=(1,1)$ and $K_\lambda(z)$ by the recurrence

$$K_1(z)=0,$$

$$K_{\lambda+1}(z)=[T_\lambda(z)+K_\lambda(z)][H(z)-\sum_{1\leqslant r<\lambda} T_r(z)]$$
$$+T_\lambda(z)[Y(z)-\sum_{1\leqslant r<\lambda}(Y_r(z)+K_r(z))].$$

Using the expansions (29) and (30), an induction on $\lambda$ leads to

$$K_\lambda(z)=\kappa_\lambda+\sum_{k\geqslant 1}\kappa_{k,\lambda}(\eta-z)^{k/2},$$

where

$$\kappa_1=0,$$

$$\kappa_{i+1}=[f_i(\eta,a)+\kappa_i][a-\sum_{1\leqslant r<i} f_r(\eta,a)]$$
$$+f_i(\eta,a)\left[\frac{\mu}{1-\gamma}-\sum_{1\leqslant r<i}\left(\frac{(1-\gamma)\mu_r+\mu\gamma_r}{1-\gamma}+\kappa_r\right)\right],\quad i\geqslant 1.$$

The coefficients in the expansion of $B_\lambda(z)$ around $\eta$ are given by (24) and (25); thus,

$$B_\lambda(z)=\gamma_\lambda^{(1)}+\sum_{k\geqslant 1}\beta_{k,\lambda}^{(1)}(\eta-z)^{k/2},$$

where $\gamma_\lambda^{(1)}$ and $\beta_{k,\lambda}^{(1)}$ satisfy the same recurrences as $\gamma_\lambda$ and $\beta_{k,\lambda}$ in Section 3.3, respectively. By (31), we further find

$$\Delta(z)=\frac{\sum_{\lambda\geqslant 1}K_\lambda(z)}{1-\sum_{\lambda\geqslant 1}B_\lambda(z)},$$

and therefore the desired evaluation of $\Delta(z)$ around $\eta$. We obtain,

$$\Delta(z)=-(\eta-z)^{-1/2}\frac{\sum_{\lambda\geqslant 1}\kappa_\lambda}{\sum_{\lambda\geqslant 1}\beta_{1,\lambda}^{(1)}}+\text{terms}\,(\eta-z)^{k/2},\quad k\geqslant 0.$$

Since $\sum_{\lambda\geqslant 1}\kappa_\lambda=2.113\,921\,889\,537\ldots$ and $\sum_{\lambda\geqslant 1}\beta_{1,\lambda}^{(1)}=-2.195\,944\,277\,826\ldots$ (see

Section 2.3), formula (1) and the Darboux-Pólya theorem leads to the following result.

**Theorem 11.** *The average right path length $\underline{RPL}(n)$ of a random leftist tree with $n$ leaves is asymptotically given by*

$$\underline{RPL}(n) \sim 3.597\,060\,138\,096\ldots n. \qquad \square$$

Note that the corresponding value for random extended binary trees with $n$ leaves is $\underline{RPL}(n) = \underline{LPL}(n) \sim 2n$ (see [3]).

## 5. Final remarks

The results derived in this paper are valid only in the case of random leftist trees; in practice, the leftist trees with $n$ leaves representing priority queues with $(n-1)$ keys are not equally likely. Thus, the presented results are primarily of theoretical interest and the exact asymptotic behaviour of the discussed parameters remains an unsolved question.

Naturally, there are further familiar parameters which play an important part in the analysis of particular algorithms operating on tree structures and which satisfy the definition of the additive weight defined in this paper. For example, such a parameter is the number of nodes with null left (right) subtrees defined by $|\{v \in I \,|\, d(v, a_v) = 1\}|$ ($|\{v \in I \,|\, d(v, b_v) = 1\}|$), where $a_v$ ($b_v$) denotes the leftmost (rightmost) leaf of the subtree with root $v$. This quantity appears in the analysis of algorithms traversing threaded binary trees and is characterized by the weight with the parameters $g(\lambda, n) \equiv 0, (c_1, c_2) = (1, 1)$ and $(\Phi_1(n), \Phi_2(n)) = (\delta_{n,1}, 0)$ $((\Phi_1(n), \Phi_2(n)) = (0, \delta_{n,1}))$. In the case of random leftist trees, its average behaviour can also be computed by the procedure discussed in this paper.

Although the presented results are substantially similar to those of simply generated families of trees (see [4], [8]), the family of leftist trees cannot be defined by restriction on the set of node degrees; in other words, there is no regular function $\Theta(y) = 1 + \sum_{\lambda \geq 1} c_\lambda y^\lambda$, $c_\lambda \in \{0, 1\}$, such that the enumerator $H(z)$ satisfies a functional equation $H(z) = z\Theta(H(z))$. This can directly be verified as follows: Part (g), (h) and (i) of Theorem 5 in [4] implies that the following fact holds for an *arbitrary* family of simply generated trees:

$$\underline{IPL}(n) \sim \tfrac{1}{2}\underline{EPL}(n)\,[\underline{RBL}(n) - 1].$$

But, in the case of random leftist trees, we find by formula (2a), Theorem 5 and Theorem 6

$$\frac{\underline{IPL}(n)}{\underline{EPL}(n)} \sim 1 \quad \text{and} \quad \tfrac{1}{2}[\underline{RBL}(n)-1] \sim 3.461\,496\ldots .$$

## Appendix

The following tables summarize the first exact and asymptotic values of the expected values of the parameters discussed in this paper. In each case, the exact values were computed by means of Lemma 1 and Theorem 2 as follows: Passing to the coefficients of $T_\lambda(z)$, Lemma 1 immediately leads to

$$t_\lambda(n)=\delta_{\lambda,1} \quad , \quad t_1(n)=\delta_{n,1}$$

and for $\lambda \geqslant 1$ to

$$t_{\lambda+1}(n)=\sum_{1\leqslant i<n} t_\lambda(i) \sum_{\lambda\leqslant r\leqslant \lfloor ld(2(n-i))\rfloor} t_r(n-i).$$

Passing now to the coefficients of $Y_\lambda(z)$, Theorem 2 yields

$$t_\lambda(1)\,\underline{w}^\lambda(1)=\delta_{\lambda,1}\,g(1,1) \quad , \quad t_1(n)\,\underline{w}_1(n)=\delta_{n,1}\,g(1,1)$$

and for $\lambda \geqslant 1$

$$\begin{aligned} t_{\lambda+1}(n)\,\underline{w}_{\lambda+1}(n) &= t_{\lambda+1}(n)\,g(\lambda+1,n) \\ &\quad + \sum_{1\leqslant i<n} [c_1\,\underline{w}_\lambda(i)+\Phi_1(i)]\,t_\lambda(i) \sum_{\lambda\leqslant r\leqslant \lfloor ld(2(n-i))\rfloor} t_r(n-i) \\ &\quad + \sum_{1\leqslant i<n} t_\lambda(i) \sum_{\lambda\leqslant r\leqslant \lfloor ld(2(n-i))\rfloor} [c_2\,\underline{w}_r(n-i)+\Phi_2(n-i)]\,t_r(n-i). \end{aligned}$$

Thus, we have a recurrence for $t_\lambda(n)$ and $\underline{w}_\lambda(n)$. Now, $t(n)$ and $\underline{w}(n)$ can be computed by the relations $t(n)=\sum_{\lambda\geqslant 1} t_\lambda(n)$ and $\underline{w}(n)=\sum_{\lambda\geqslant 1} t_\lambda(n)\,\underline{w}_\lambda(n)/t(n)$ (see remark following Lemma 1 and formulas (14) and (15)).

Table 3
The exact (ex.) and asymptotic (as.) values of the average external (free) path length $\underline{EPL}(n)$ ($\underline{EFPL}(n)$) and of the left (right) path length $\underline{LPL}(n)$ ($\underline{RPL}(n)$) of a random leftist tree with $n$ leaves.

| $n$ | $\underline{EPL}(n)$ | | $\underline{EFPL}(n)$ | | $\underline{LPL}(n)$ | | $\underline{RPL}(n)$ | |
|---|---|---|---|---|---|---|---|---|
| | ex. | as. | ex. | as. | ex. | as. | ex. | as. |
| 1 | 0.000 | 1.607 | 0.000 | 0.804 | 0.000 | 1.346 | 0.000 | 3.597 |
| 2 | 2.000 | 4.546 | 2.000 | 4.546 | 1.000 | 2.693 | 1.000 | 7.194 |
| 3 | 5.000 | 8.351 | 8.000 | 12.527 | 2.000 | 4.039 | 3.000 | 10.791 |
| 4 | 8.500 | 12.857 | 19.500 | 25.715 | 3.500 | 5.386 | 5.000 | 14.388 |
| 5 | 12.750 | 17.969 | 37.000 | 44.922 | 4.750 | 6.732 | 7.500 | 17.985 |
| 6 | 17.625 | 23.620 | 61.375 | 70.861 | 6.000 | 8.079 | 10.375 | 21.582 |
| 7 | 23.000 | 29.765 | 93.412 | 104.178 | 7.294 | 9.425 | 13.353 | 25.179 |
| 8 | 28.789 | 36.366 | 133.789 | 145.464 | 8.632 | 10.771 | 16.368 | 28.776 |
| 9 | 35.011 | 43.394 | 183.034 | 195.271 | 9.966 | 12.118 | 19.483 | 32.374 |
| 10 | 41.640 | 50.823 | 241.680 | 254.115 | 11.300 | 13.464 | 22.670 | 35.971 |
| 20 | 126.637 | 143.749 | 1 446.488 | 1 437.494 | 24.723 | 26.929 | 56.321 | 71.941 |
| 30 | 239.258 | 264.084 | 4 041.142 | 3 961.267 | 38.167 | 40.393 | 91.338 | 107.912 |
| 40 | 374.150 | 406.585 | 8 337.430 | 8 131.693 | 51.622 | 53.857 | 126.787 | 143.8[illegible] |
| 50 | 528.240 | 568.219 | 14 595.209 | 14 205.483 | 65.080 | 67.321 | 162.428 | 179.[illegible] |
| 60 | 699.465 | 746.944 | 23 042.265 | 22 408.312 | 78.540 | 80.786 | 198.172 | 215.824 |
| 70 | 886.313 | 941.257 | 33 883.953 | 32 943.982 | 92.001 | 94.250 | 233.976 | 251.794 |
| 80 | 1 087.614 | 1 149.995 | 47 308.776 | 45 999.803 | 105.463 | 107.714 | 269.820 | 287.765 |
| 90 | 1 302.425 | 1 372.223 | 63 491.972 | 61 750.042 | 118.926 | 121.179 | 305.691 | 323.735 |
| 100 | 1 529.971 | 1 607.167 | 82 597.979 | 80 358.348 | 132.389 | 134.643 | 341.581 | 359.706 |
| 120 | 2 020.729 | 2 112.676 | 130 192.429 | 126 760.555 | 159.315 | 161.572 | 413.399 | 431.647 |

Table 4
The exact values of the average left (right) branch length $\underline{LBL}(n)$ ($\underline{RBL}(n)$) of a random leftist tree with $n$ leaves.

| $n$ | $\underline{LBL}(n)$ | $\underline{RBL}(n)$ | $n$ | $\underline{LBL}(n)$ | $\underline{RBL}(n)$ |
|---|---|---|---|---|---|
| 1 | 0.000 000 | 0.000 000 | 20 | 1.725 118 | 5.968 273 |
| 2 | 1.000 000 | 1.000 000 | 30 | 1.754 420 | 6.525 355 |
| 3 | 1.000 000 | 2.000 000 | 40 | 1.769 116 | 6.835 352 |
| 4 | 1.500 000 | 2.500 000 | 50 | 1.777 953 | 7.032 792 |

Table 4 contd.

| | | |
|---|---|---|
| 5 | 1.500 000 | 3.000 000 |
| 6 | 1.500 000 | 3.500 000 |
| 7 | 1.529 412 | 3.882 353 |
| 8 | 1.578 947 | 4.184 211 |
| 9 | 1.609 195 | 4.448 276 |
| 10 | 1.630 542 | 4.679 803 |
| 60 | 1.783 851 | 7.169 558 |
| 70 | 1.788 067 | 7.269 894 |
| 80 | 1.791 231 | 7.346 646 |
| 90 | 1.793 693 | 7.407 255 |
| 100 | 1.795 663 | 7.456 329 |
| 120 | 1.798 862 | 7.530 938 |
| ↓ | ↓ | ↓ |
| ∞ | 1.813 410... | 7.922 992... |

Table 5
The exact (ex.) and asymptotic (as.) values of the average number $\underline{EP}_r(n)$ of root-free paths and of the average number of leaves $\underline{L}_r(n)$ ($\underline{L}_l(n)$) in the right (left) subtree of the root of a random leftist tree with $n$ leaves.

| $n$ | $\underline{EP}_r(n)$ | | $\underline{L}_r(n)$ | | $\underline{L}_l(n)$ | |
|---|---|---|---|---|---|---|
| | ex. | as. | ex. | as. | ex. | as. |
| 1 | 0.000 | 0.5 | 0.000 | 0.754 | 0.000 | 0.245 |
| 2 | 0.000 | 2 | 1.000 | 1.509 | 1.000 | 0.491 |
| 3 | 1.000 | 4.5 | 2.000 | 2.263 | 1.000 | 0.737 |
| 4 | 2.500 | 8 | 2.500 | 3.018 | 1.500 | 0.982 |
| 5 | 5.000 | 12.5 | 3.250 | 3.772 | 1.750 | 1.228 |
| 6 | 8.375 | 18 | 4.125 | 4.527 | 1.875 | 1.473 |
| 7 | 12.529 | 24.5 | 4.882 | 5.281 | 2.118 | 1.719 |
| 8 | 17.447 | 32 | 5.605 | 6.036 | 2.395 | 1.964 |
| 9 | 23.241 | 40.5 | 6.356 | 6.790 | 2.644 | 2.210 |
| 10 | 29.906 | 50 | 7.113 | 7.545 | 2.887 | 2.455 |
| 20 | 145.223 | 200 | 14.651 | 15.089 | 5.349 | 4.911 |
| 30 | 351.294 | 450 | 22.196 | 22.634 | 7.804 | 7.366 |
| 40 | 649.867 | 800 | 29.741 | 30.178 | 10.259 | 9.822 |
| 50 | 1 041.974 | 1 250 | 37.285 | 37.723 | 12.715 | 12.277 |
| 60 | 1 528.310 | 1 800 | 44.830 | 45.267 | 15.170 | 14.733 |
| 70 | 2 109.384 | 2 450 | 52.374 | 52.812 | 17.626 | 17.188 |
| 80 | 2 785.591 | 3 200 | 59.919 | 60.356 | 20.081 | 19.644 |
| 90 | 3 557.246 | 4 050 | 67.463 | 67.901 | 22.537 | 22.099 |
| 100 | 4 424.613 | 5 000 | 75.008 | 75.445 | 24.992 | 24.555 |
| 120 | 6 447.335 | 7 200 | 90.097 | 90.534 | 29.903 | 29.466 |

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