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COMPUTATION OF THE SOLUTIONS OF THE DIOPHANTINE EQUATION $x^3 + dy^3 = 1$

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1. Introduction.

Let $K(d^{1/3})$ be the algebraic number field formed by adjoining $d^{1/3}$, where d is a positive integer, and not a perfect cube, to the rationals. The complete solution of the Diophantine equation

$$(1) \qquad x^3 + dy^3 = 1$$

is given in the following

Theorem. (Delone - Nagell [5,6]) The equation (1) has at most one solution in integers x_1, y_1 where $x_1 y_1 \neq 0$. If (x_1, y_1) is a solution of (1), then $x_1 + y_1 d^{1/3}$ is either the fundamental unit of $K(d^{1/3})$ or its square; the latter happens only for $d = 19, 20, 28$.

The problem, then, of finding a solution of (1) is equivalent to that of finding a fundamental unit in $K(d^{1/3})$. This can be done by using the Algorithm of Voronoi (see Delone and Faddeev [4]). This algorithm was used by Beach, Zarnke, and Williams [1] to find all the solutions of (1) for $2 \leq d \leq 1000$. Bernstein [2] used the Jacobi-Perron algorithm to find solutions of (1) with $2 \leq d \leq 1000$.

In this paper, we give all solutions of (1) for all values of d such that $2 \leq d \leq 15000$.

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2. Computer Programme.

The computer programme used to calculate the fundamental unit ε_o $(0 < \varepsilon_o < 1)$ of $K(d^{1/3})$ is described in [1]. Let

$$\varepsilon_o = (u + vd^{1/3} + wd^{2/3})/t ,$$

where u, v, w, t are integers. The values of $|u|, |v|, |w|$ seem to be approximately $10^{p/4}$, where p is the period length of Voronoi's algorithm for $d^{1/3}$. For example, when $d^{1/3} = 951^{1/3}$, $p = 1352$ and $|u| \approx 8 \cdot 10^{330}$, $|v| \approx 2 \cdot 10^{330}$, $|w| \approx 9 \cdot 10^{329}$. For larger values of d, the period lengths and consequently the values of $|u|, |v|, |w|$, become quite large (e.g. when $d = 14153$, $p = 18196$). In order to circumvent the time-consuming and difficult process of finding the exact multi-precise values of u, v, w, the programme was designed to determine the residue of each of these numbers modulo a large prime q. (We used $q = 100000007$).

If, for some value of d,

$$w \equiv 0 \pmod q ,$$

the exact unit was calculated in order to determine whether $w = 0$. It turned out that, in the range in which the programme was working, $w = 0$ whenever $w \equiv 0 \pmod q$.

Since Voronoi's algorithm is quite long and complicated, we included a preliminary sieving routine which excluded several values of d for which (1) could not have a non-trivial solution. These values of d, with the exception of 2, 9, 17, 20, are those integers which do not have at least one prime divisor congruent to 1 modulo 3 (see Cohn [3]). About 30.4 per cent of all d values considered were eliminated by this process.

Since $K(d^{1/3})$ is the same field as $K(c^{1/3})$, when $d = r^3c$, all values of d, which contained a factor which was a perfect integer cube, were eliminated. 16.8 per cent of all positive values of $d \leq 15000$ contain a perfect cube factor.

3. Results.

In Table 1, we give all the non-trivial solutions (x,y) of (1) for all values of d such that $2 \leq d \leq 15400$. An IBM/360-65 computer required 200 minutes of CPU time to produce this table.

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TABLE 1

d	x	y
2	-1	1
7	2	-1
9	-2	1
17	18	-7
19	-8	3
20	-19	7
26	3	-1
28	-3	1
37	10	-3
43	-7	2
63	4	-1
65	-4	1
91	9	-2
124	5	-1
126	-5	1
182	-17	3
215	6	-1
217	-6	1
254	19	-3
342	7	-1
344	-7	1
422	-15	2
511	8	-1
513	-8	1
614	17	-2
635	361	-42
651	-26	3
728	9	-1
730	-9	1
813	28	-3
999	10	-1
1001	-10	1
1330	11	-1
1332	-11	1
1521	-23	2
1588	-35	3
1657	-71	6
1727	12	-1
1729	-12	1

(continued on next page)

TABLE 1

d	x	y
1801	73	-6
1876	37	-3
1953	25	-2
2196	13	-1
2198	-13	1
2743	14	-1
2745	-14	1
3155	-44	3
3374	15	-1
3376	-15	1
3605	46	-3
3724	-31	2
3907	-63	4
4095	16	-1
4097	-16	1
4291	65	-4
4492	33	-2
4912	17	-1
4914	-17	1
5080	361	-21
5514	-53	3
5831	18	-1
5833	-18	1
6162	55	-3
6858	19	-1
6860	-19	1
7415	-39	2
7999	20	-1
8001	-20	1
8615	41	-2
8827	-62	3
9260	21	-1
9262	-21	1
9709	64	-3
10647	22	-1
10649	-22	1
12166	23	-1
12168	-23	1
12978	-47	2

(continued on next page)

TABLE 1

d	x	y
13256	-71	3
13538	-143	6
13823	24	-1
13825	-24	1
14114	145	-6
14408	73	-3
14706	49	-2
15253	-124	5

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