Chapter 1

Number Representations and Dragon Curves

[Written with Chandler Davis. Originally published in the Journal of Recreational Mathematics **3** *(1970), 66–81, 133–149.]*

1. Introduction

Take a long strip of paper and fold it in half; then fold the result in half again, several more times, as shown in Figure 1. When the paper is opened up again, it displays an interesting pattern of creases.

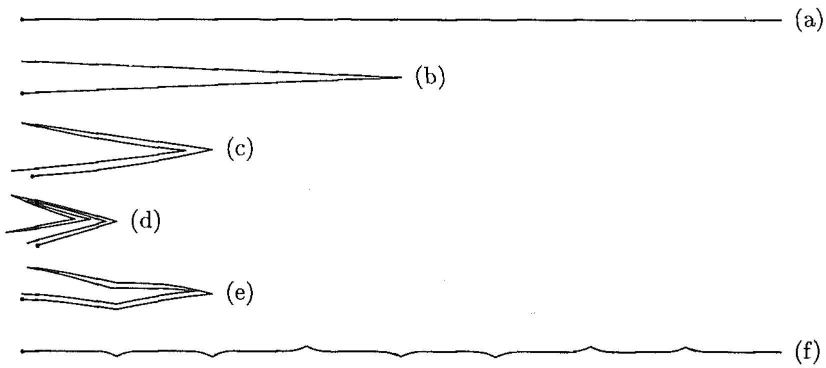


FIGURE 1. Folding a strip of paper.

If we write D for a crease that makes the paper dip downward (a "valley fold"), and U for one that makes an upward hump (a "mountain fold"), Figure 1 shows that after n folds the creases form the following patterns:

$$\begin{array}{ll} n=1 & D \\ n=2 & DDU \\ n=3 & DDUDDUU \end{array}$$

It is easy to see that there will be $2^n - 1$ creases after n folding operations have been performed, since the paper has been divided into 2^n areas.

Let S_n be the sequence of $2^n - 1$ Ds and Us that is obtained after n folding operations. One way to analyze the sequences S_1, S_2, S_3, ... is to observe that S_{n+1} always begins with S_n, since we can imagine starting with paper twice as long when we want to do $n+1$ folds. In the same way we can see that the sequence S_{n+1} ends with the sequence S_n in backwards order, with U and D interchanged, since the last n folds act in essentially the same way on the first and last halves of the paper. Thus we know that S_4 must begin with $S_3 = DDUDDUU$, and it must end with S_3 backwards (namely $UUDDUDD$) but with U and D swapped (namely $DDUDDUU$). If we let $\overline{S}$ denote the sequence obtained from a sequence S by writing it backwards and interchanging U and D, we therefore get the simple formula

$$S_{n+1} \;=\; S_n D \overline{S_n}. \tag{1.1}$$

(The middle letter, which comes from the first fold, is always D.) This rule defines S_n for any n, starting with the "empty" sequence S_0. Notice that for any sequences S and T we have

$$\overline{\overline{S}} = S \qquad \text{and} \qquad \overline{ST} = \overline{T}\,\overline{S};$$

therefore

$$\overline{S_{n+1}} \;=\; S_n U \overline{S_n}. \tag{1.2}$$

In other words, $\overline{S_n}$ turns out to be the same as S_n, except that the middle letter is changed from D to U. This is the sequence of creases we get if we turn the paper end-for-end.

There is yet another way to obtain the sequence S_{n+1} from S_n, if we concentrate our attention on the last fold instead of the first: The 2nd, 4th, 6th, ... creases in S_{n+1} are evidently the same as the creases of S_n. Furthermore, the 1st, 3rd, 5th, ... creases are alternately D, U, D, U, ...; this is easily seen from Figure 1(e), since the odd-numbered creases must alternate regardless of the pattern of the even-numbered ones. Thus $S_n = a_1 a_2 a_3 \ldots a_m$ implies that

$$S_{n+1} = D a_1 U a_2 D a_3 U \ldots D a_m U. \tag{1.3}$$

Since S_{n+1} begins with S_n it makes sense to talk of the infinite sequence S_∞ that arises in the limit. Rule (1.3) gives us a quick way to write down as much of S_∞ as we please. First we write alternating Ds and Us, leaving spaces between them:

$$D \quad U \quad D \quad U \quad D \quad U \quad D \quad U \quad D \quad U \quad \ldots$$

Then, using our left hand to point and our right hand to write, and with

7. Conclusion

We have seen that paper folding leads to some curves with remarkable properties; these properties are intimately related to number systems for integers and for lattices of integers in the plane.

Are there 3-dimensional "dragon curves" that have aesthetic properties comparable to the 2-dimensional ones considered here? One way to generalize what we have done above is to let $\delta(n)$ be a vector-valued function satisfying the relation

$$\delta(2n) = A\delta(n)$$

for some appropriate matrix A. We have not been able to discover any 3-dimensional generalization of any particular interest, although it seems not unlikely that some crystal structure possesses paths of comparable beauty.

The authors thank Donald Coxeter for helpful comments. The research of Chandler Davis was supported in part by the National Research Council of Canada.

References

[1] Martin Gardner, "Mathematical Games," *Scientific American* **216**, 3 (March 1967), 124–125; **216**, 4 (April 1967), 118–120; **217**, 1 (July 1967), 115. Reprinted in *Mathematical Magic Show* (New York: Knopf, 1977), Chapter 15, Problem 7.

[2] Walter Helbig Gottschalk and Gustav Arnold Hedlund, *Topological Dynamics*, American Mathematical Society Colloquium Publication 36 (Providence, Rhode Island: American Mathematical Society, 1955).

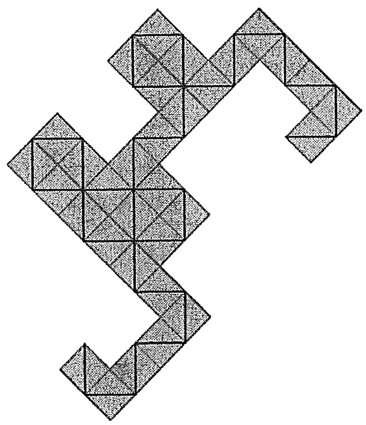
[3] W. G. Harter and J. E. Heighway, personal communication.

[4] Donald E. Knuth, *Seminumerical Algorithms*, Volume 2 of *The Art of Computer Programming* (Reading, Massachusetts: Addison–Wesley, 1969), Section 4.1.

Addendum

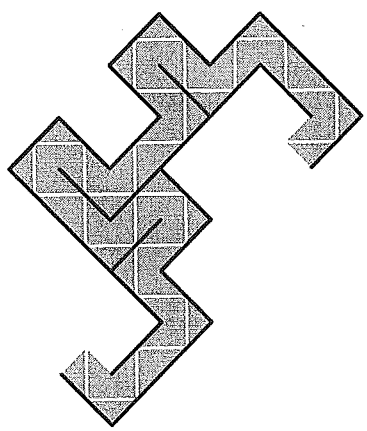
Another elementary way to visualize the dragon curve and its cousins occurred to me in 2009: We can "fatten up" the zigzag path by enclosing each edge — or | within a diamond-shaped tile ◇ or ◇, so that each unit segment becomes the diagonal of a tile instead of an actual edge.

These tiles fit together nicely because the path always changes direction by $\pm 90°$; for example, Figure 7 becomes



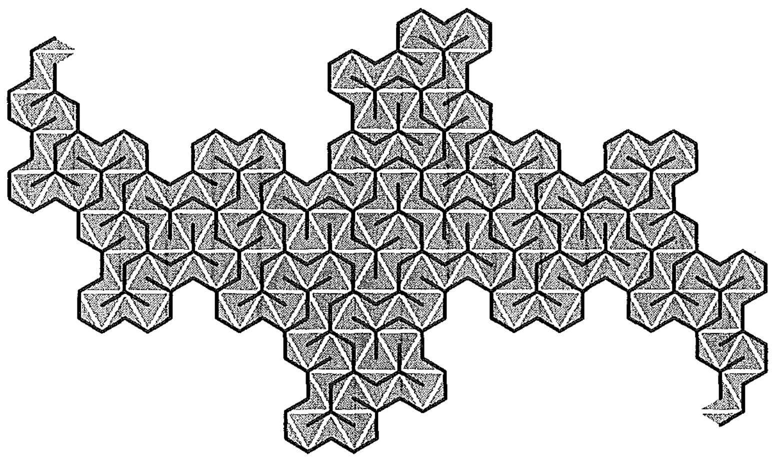
when embellished in this way. The dragon design becomes a dragon polyomino!

Furthermore, we can place "walls" of length $\sqrt{2}$ wherever two such tiles join at a bend of the path. The resulting pattern can be illustrated as follows, showing walls in black and the original path in white:



Notice that the zigzag path bounces off the walls in this interpretation. If the walls were mirrors, a beam of light would perfectly trace out the dragon design as it passes through this system of walls, which I like to call the *dragon labyrinth* of order 5.

The ter-dragon can be fattened up with lozenge-shaped tiles in a similar way, although the mirror-reflection idea no longer prevails. Here's the ter-dragon labyrinth of order 4:



While preparing Figure 2, which opens up the dragon-sequence folds to angles of 100° at each bend, I noticed in 1969 that 95°-angle folds would lead to paths that cross themselves. For example, the path obtained from S_{10} will interfere with itself just before points 447 and 703; and if we look further, 95° bends applied to S_{12} will yield a path that crosses itself quite dramatically before and after points 1787 and 2807.

Consider the continuous curve $\delta(t)$ indicated in (3.14), and allow the angle θ to vary. As θ decreases from 180° it reaches a critical value θ_c where $\delta(t)$ first touches itself (that is, first ceases to be a one-to-one function from the positive real numbers into the complex plane). This value θ_c seems to lie between 95° and 96°, but I don't know how to calculate it. The most troublesome crossing points appear to lie near $7\cdot 2^n$ and $11\cdot 2^n$.

Number theorists will recognize the sequence $d(n)$ in (3.3) as the Jacobi symbol $(\frac{-1}{n})$. We also have $d(n) = \operatorname{sign}(G(n) - G(n-1))$, where $G(n)$ is the nth element of "Gray binary code," namely $n \oplus \lfloor n/2 \rfloor$; this fact was noted by George P. Darwin in a letter to Martin Gardner dated 26 April 1967. Indeed, if we let $\nabla(n) = G(n) - G(n-1)$, it is easy to see that $\nabla(2n) = 2\nabla(n)$ and $\nabla(2n+1) = (-1)^n$. [Gray binary code is discussed in Section 7.2.1.1 of *The Art of Computer Programming*; the Jacobi symbol is discussed, for example, in exercise 4.5.4–23 of that work.]

The "Morse–Hedlund" sequence discussed in this chapter is now commonly known as the Thue–Morse sequence, because it appeared in two classic early works: Axel Thue, "Über die gegenseitige Lage gleicher Teile gewisser Zeichenreihen," *Skrifter udgivne af Videnskabs-Selskabet i Christiania*, Mathematisk-Naturvidenskabelig Klasse (1912), No. 1, 1–65, §7; Harold Marston Morse, "Recurrent geodesics on a surface of negative curvature," *Transactions of the American Mathematical Society* **22** (1921), 84–100, §14.

The Dutch mathematician F. M. Dekking wrote to Martin Gardner in the summer of 1975, describing many new kinds of dragon-like curves. If $S = a_1a_2 \ldots a_{s-1}$ and $T = b_1b_2 \ldots b_{t-1}$ are any sequences of folds, where each element a_i of S and each element b_j of T is either D or U, he defined the "folding product" $S * T = c_1c_2 \ldots c_{st-1}$ by the rule

$$S * T \;=\; \begin{cases} Sb_1\overline{S}b_2S \ldots Sb_{t-1}\overline{S}, & \text{if } t \text{ is even;} \\ Sb_1\overline{S}b_2S \ldots \overline{S}b_{t-1}S, & \text{if } t \text{ is odd.} \end{cases}$$

He observed that this definition, which generalizes formulas (1.1), (4.1), and (5.1), is *associative*; in other words, we have $R * (S * T) = (R * S) * T$ for any sequences R, S, and T. Therefore it makes sense to consider n-ary folding products $S_1 * S_2 * \cdots * S_n$, as well as infinite folding products $S_1 * S_2 * S_3 * \cdots$. Notice that

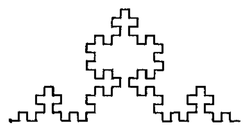
$$\overline{S * T} \;=\; \begin{cases} S * \overline{T}, & \text{if } t \text{ is even;} \\ \overline{S} * \overline{T}, & \text{if } t \text{ is odd.} \end{cases}$$

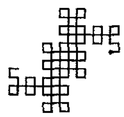
If we define the folding powers

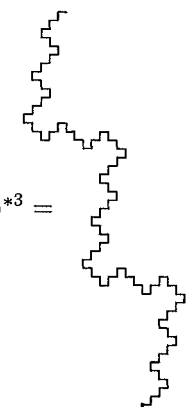
$$S^{*n} \;=\; \overbrace{S * \cdots * S}^{n \text{ times}},$$

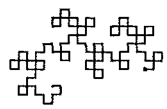
then the dragon sequence of order n is $D * \cdots * D = D^{*n}$, and the ter-dragon sequence of order n is $DU * \cdots * DU = (DU)^{*n}$. Furthermore, the alternate dragon and ter-dragon sequences of order $2n$ are respectively $(DUU)^{*n}$ and $(DUUDUDDU)^{*n}$ in this notation.

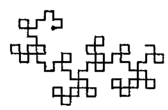
Folding powers produce many *new* species of dragons. For example, let's consider sequences S of four folds; these are the folding patterns that divide a strip of paper into five equal parts, so that S^{*n} will yield 5^n equal parts. If we change all Ds to Us and vice versa, everything is simply reflected as in a mirror; hence we can restrict consideration to the eight cases where S begins with D. Suppose we open all folds to 90°. Then the sequence $DDDD$ intersects itself; and when $S = DDDU$ or $DUUU = \overline{DDDU}$, the sequence $S^{*2} = S * S$ intersects itself. But the other five cases yield arbitrarily long folding powers S^{*n} that never repeat any edges, for any n, illustrated here for $n = 3$:

$(DUUD)^{*3} =$ 

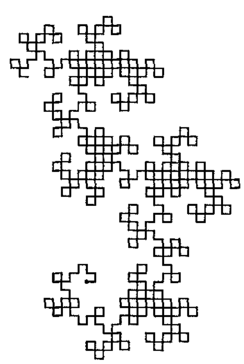
$(DDUU)^{*3} =$ 

$(DUDU)^{*3} =$ 

$(DUDD)^{*3} =$ 

$(DDUD)^{*3} =$ 

(In each case the origin point is indicated by a small dot.) The first and last examples leave the origin so fast, their points never touch. But the other three examples are *plane-filling*, in the sense that all of the edges in an $m \times m$ subgrid are covered somewhere within S^{*n}, when n is sufficiently large (depending on m). For example, $(DDUU)^{*3}$ covers several 2×2 subgrids, and $(DDUD)^{*3}$ covers several 1×1's; and

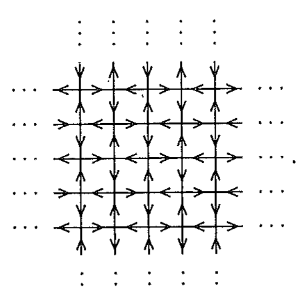
$(DDUD)^{*4} =$ 

covers several 3×3's.

Let's say that a folding sequence S is *self-avoiding* if its iterated folding power $S^{*\infty}$ never traverses the same edge twice, always assuming $90°$ folds. Furthermore, a self-avoiding sequence can also be *plane-filling* in the sense above, covering arbitrarily large subgrids. Finally, Dekking called S *perfect* when the four path $(S^{*\infty}, iS^{*\infty}, -S^{*\infty}, -iS^{*\infty})$ cover every grid edge exactly once. Thus, the dragon sequence $S = D$ is perfect, and so are the sequences $DDUU$, $DUDD$, $DDUD$. Dekking conjectured in 1975 that $S * T$ is perfect when S and T are perfect.

A sequence S can be plane-filling without being perfect. Indeed, we have seen that this is true for the alternate dragon curve, when $S = DUU$; in that case $S^{*\infty}$ covers just 1/8 of the plane, not 1/4, because $S^{*\infty} \cup (iS^{*\infty}) \cup (-S^{*\infty}) \cup (-iS^{*\infty})$ only covers about half of the edges (see Figure 10). Since $DUU = D * U$ while the simple sequences D and U are both perfect, we should revise the conjecture, limiting it to cases where S and T both begin with D. In that form, Dekking was able to prove his conjecture several years later, by finding nice ways to characterize exactly when a sequence is self-avoiding, plane-filling, and/or perfect.

First, he showed that S is self-avoiding if and only if $S * DDD$ has no repeated edges; indeed, he showed more generally that if $S * DDD$ and T have no repeated edges, then $S * T$ has no repeats. He did this by first observing that grid paths in which 90° turns occur after every step are equivalent to paths in the infinite directed graph



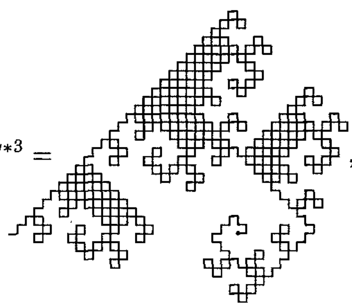
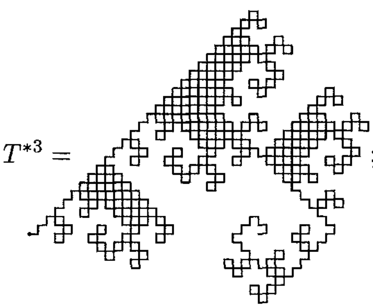
[See F. M. Dekking, "Iterated paperfolding and planefilling curves," Report 8126 (Nijmegen, The Netherlands: Katholieke Universiteit, Mathematisch Instituut, 1981), 16 pages.]

Second, he proved that a self-avoiding S with $s - 1$ folds is plane-filling if and only if its grid path (which has length s) ends at a point $z = a + bi$ for which $a^2 + b^2 = s$. (For example, the five self-avoiding examples shown earlier for $s = 5$ end at the respective points 3, $1 + 2i$, $1 + 2i$, $-1 + 2i$, and $3 + 2i$; hence only the middle three are plane-filling.)

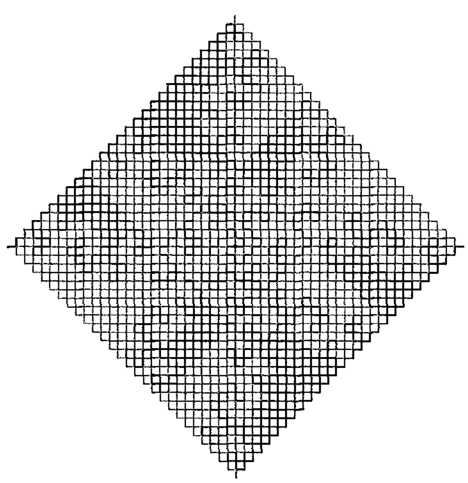
Third, he discovered the remarkable fact that a plane-filling S is perfect if and only if its grid path does not go through the edge from $z + zi - 1 - i$ to $z + zi - i$. And he constructed perfect sequences S of length s whenever $s > 1$ can be written as the sum of two squares, by defining an appropriate s-arc subgraph of the infinite digraph above and using the fact that a balanced digraph always contains an Eulerian trail.

His conjecture about the product of perfect sequences follows from these facts, in a stronger form: *If S and T are plane-filling sequences that both begin with D, then $S * T$ is perfect if and only if either S is perfect or T is perfect.*

The possibilities are much trickier than they may seem at first glance. Consider, for example, the sequences $S = DDUDDUDU$ and $T = DUDUUDUU = \overline{S}$, which define paths of length $s = 9$ to the points $z = -3$ and $z = 3$, respectively. We have

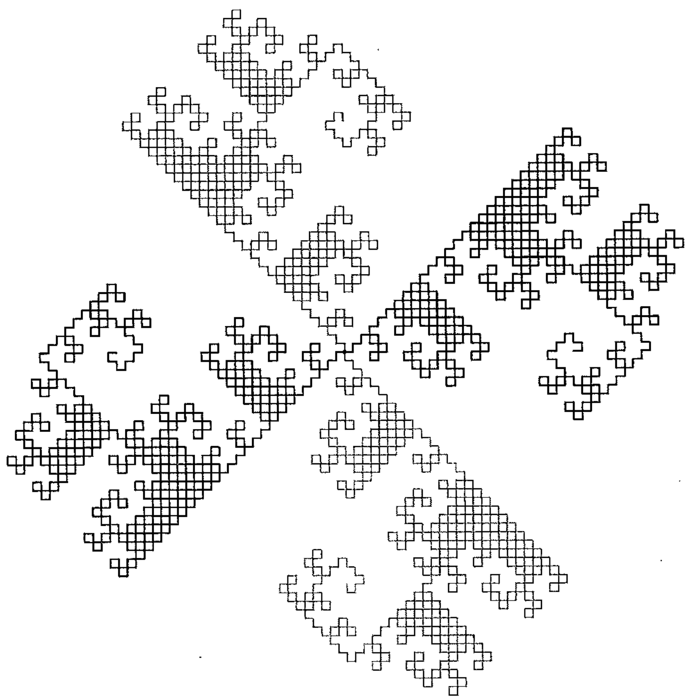
$S^{*3} =$  , $T^{*3} =$  ;

both paths are the same, except for the starting point. Yet the starting point makes a huge difference, because S is perfect but T is not! If we draw S^{*3} and $-S^{*3}$ in black, together with iS^{*3} and $-iS^{*3}$ in gray, we get a beautiful design that perfectly fills a diamond-shaped region about the origin:



It's a design with a meso-American flavor, also reminiscent of meandering patterns in the "late geometric" style that was popular on Greek vases in the 8th century B.C.

But the same tactics with T^{*3} lead to a far different result:



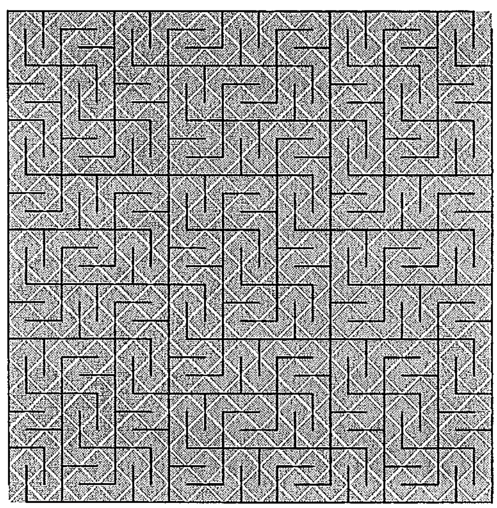
Each of the four copies of T^{*n} covers a region like S^{*n}, which we know fills 1/4 of a diamond. Hence $T^{*\infty}$ covers just 1/16 of the plane—only half as much as the alternate dragon curve, which covers 1/8.

This example is the case $m = 3$ of a general construction $S_m = D(DU)^1 D(DU)^2 \ldots D(DU)^{m-1}$ that yields similar (but ever more intricate) folding paths of length $s = m^2$ that lead from 0 to $i^{m-1}m^2$. The reader is encouraged to try drawing S_5^{*3} (with computer help).

The simplicity of Dekking's characterizations makes it easy to count exactly how many s-folds of various types are possible, for small s. Suppose A_s of the 2^{s-2} possibilities beginning with D are self-avoiding but not plane-filling; B_s are plane-filling but not perfect; and C_s are perfect. Then we have:

$s =$	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
$A_s =$	0	1	1	2	7	10	15	33	45	93	186	300	530	825	1561	2722	4685	7419	13563
$B_s =$	0	0	1	0	0	0	0	2	2	0	0	2	0	0	8	0	4	0	12
$C_s =$	1	0	1	3	0	0	6	3	20	0	0	29	0	0	56	101	108	0	392

In his letters of 1975 to Martin Gardner, Dekking pointed out that the sequence $S = DUUUDDDU$, which is the simplest perfect pattern for $s = 9$, actually generates the famous space-filling curve of Guiseppe Peano ["Sur une courbe qui remplit toute une aire plane," *Mathematische Annalen* **36** (1890), 157–160], which was the first-ever construction of a continuous function from the unit interval $[0\,..\,1]$ to the unit square $[0\,..\,1] \times [0\,..\,1]$. Indeed, S^{*2} in this case generates a path that corresponds via our polyomino construction to



(but rotated by 45°), which we may call "Peano's labyrinth of order 3." [See Hans Sagan, *Space-Filling Curves* (New York: Springer, 1994), Fig. 3.6.2.]

Dekking has not yet published the proofs of his theorems about generalized dragons, perhaps because he has felt that they are too elementary. I have recently encouraged him to communicate them because of the growing interest in this topic.

This theory suggests several more intriguing problems that I have no idea how to solve. For instance:

1) If S is a plane-filling sequence, we have seen that its iterated extension $S^{*\infty}$ might fill 1/4, 1/8, or 1/16 of the entire plane. Are other values of this overall density possible? And how can the density be computed, given S?
2) Given an infinite sequence $a_1, a_2, a_3, \ldots$ of Ds and/or Us, how can one determine the density of the plane-filling path $a_1 * a_2 * a_3 * \cdots$? What is the smallest possible value of this density? (For example, we know that the density is 1/4 when $a_j = D$ for all j; it is 1/8 when $a_j = U$ if and only if j is even. What is the density when $a_j = U$ if and only if j is, say, a perfect square, or a prime number? Notice that the density is unchanged if we change any *finite* number of the parameters a_j from D to U or vice versa.)
3) The theory of plane-filling S can be developed for folds of 60° as well as 90°, but detailed characterizations have not yet been worked out. In this case a "perfect" S would be a sequence for which the six paths $(S^{*\infty}, \omega S^{*\infty}, \omega^2 S^{*\infty}, -S^{*\infty}, -\omega S^{*\infty}, -\omega^2 S^{*\infty})$ cover every edge of the hexagonal grid exactly once. (Dekking remarked in 1975 that every perfect S that he knows for 60° folds was "balanced," in the sense that $\bar{S} = S$. He suspected that unbalanced cases might exist, but perhaps only when S involves a large number of folds.)

Many fascinating properties of generalized dragon curves certainly remain to be discovered.

Meanwhile, researchers have been developing the theory in other directions. For example, consider the number $f(x) = \sum_{n\geq 1} b_n x^n$, where $b_n = (d(n)+1)/2$ is 1 when the nth term of the dragon sequence S_∞ is D, otherwise $b_n = 0$. Then the "dragon constant," the binary fraction

$$f(\tfrac{1}{2}) = (0.b_1 b_2 b_3 \ldots)_2 = (0.1101100111001001110110001100\ldots)_2$$
$$\approx 0.85073\,61882\,01867\,26036\,77977\,60532\,06660\,44114-,$$

is known to be transcendental. In fact, $f(x)$ is transcendental when x is *any* algebraic number with $|x| < 1$; and the same is true for the sequences $\langle b_n \rangle$ that correspond to any of the generalized dragon curves considered in Theorem 5 and in problem (2) above. These results were proved by M. Mendès France and A. J. van der Poorten, "Arithmetic and analytic properties of paper folding sequences," *Bulletin of the Australian Mathematical Society* **24** (1981), 123–131.

Further interesting properties of dragon-like curves and sequences are explored in the expository paper "Folds!" by Michel Dekking, Michel Mendès France, and Alf van der Poorten, in *The Mathematical Intelligencer* **4** (1982), 130–138, 173–181, 190–195; **5**, 2 (1983), 5. In particular, they explain how the ideas relate to the rapidly developing theory of "automatic sequences." [See the book *Automatic Sequences* by Jean-Paul Allouche and Jeffrey Shallit (Cambridge University Press, 2003), for a comprehensive introduction to that subject.]

The question of "3-dimensional folding," which Chandler Davis and I mentioned briefly at the close of our original paper, has been fruitfully studied by Michel Mendès France and J. O. Shallit, "Wire bending," *Journal of Combinatorial Theory* **A50** (1989), 1–23, although not exactly in the way we had in mind.

Topological properties of the image of the continuous dragon curve $\delta(t)$ in (3.14) have been investigated by Sze-Man Ngai and Nhu Nguyen, "The Heighway dragon revisited," *Discrete and Computational Geometry* **29** (2003), 603–623.

John Heighway wrote to Martin Gardner on 30 December 1997, explaining that he had discovered the dragon curve but that William Harter had named it. I asked Harter in 2001 about his recollections, and he told an interesting story:

> The dragon curve was born in June of 1966. Jack [Heighway] came into my office (actually cubicle) and said that if you folded a $1 bill repeatedly he thought it would make a random walk or something like that. (We'd been arguing about something in Feller's book on return chances.) I was dubious but said "Let's check it out with a big piece of paper." (Those were the days when NASA could easily afford more than $1's worth of paper.) Well, it made a funny pattern alright but we couldn't really see it too clearly. So one of us thought to use tracing paper and "unfold" it indefinitely so we could record (tediously) as big a pattern as we wanted. But each time we made the next order, it just begged us to make one more!
>
> ... Lee Ponting, another summer student, should be mentioned, too, for writing the first program to plot the dragon using FORTRAN and a Calcomp.
>
> More recently, the dragon was used by Michael Crichton in a book about dragons entitled "Jurassic Park" to go along with a story line that includes a character who works on "fractals." Unfortunately, Mr. Crichton seems to have failed to notice why

the dragons were called dragons. He proceeded to print them "evolving" page after page, but upside down, that is, as dead dragons! Maybe this shows that you always get punished if you use something without attribution.

[See Michael Crichton, *Jurassic Park* (New York: Knopf, 1990), 9, 31, 83, 179, 269, 315, 363.]