Scan ASS89 Eureka etz 5 sheeks



Problems Drive

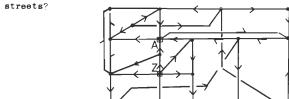
The problems in the 1974 Archimedeans versus Invariants problems drive were set by the 1973 "inning pair, Colin Vout and Martin Brown.

The answers are on the inside back cover.

1)(a) Is this a knot or not?



(b) Can you get from A to Z in this complex of one-way



(c) Below is a supposed picture of a wall tile which has progressively cracked, that is, one new crack at a time. Each new crack has its ends in the middle of a previous crack or on the edge. Could this in fact have happened?



'd) Can the following rendy-creased sheet of paper be folded up, with no 'tucking-in' needed? (Bold and dotted lines are oppositely creased.)



- 2) Let $x = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_n^{\alpha_n}$ be the expression of the integer x in prime factors such that $p_i \neq p_j$ if i = j. (i) Let d(x) be the number of divisors of x including 1 and x). Express d(x) in terms of the a's. (ii) Define S(x) = 1 if x is a perfect source, else 0. Express S(x) in terms of d(x). (iii) Let N(x) be the number of representations of x as a sum of two perfect scuares. Express N(x) in terms of S(x) and thus in terms of a sum involving only terms like d(y).
- 5) A frame is defined as a networks of sticks, or finite line segments, drawn in the plane such that each stick crosses exactly three others, alternately 'over', 'under', 'over' or vice versa. Each intersection is to involve only two sticks, one under and one over.

Find and draw all possible frames with at most seven sticks.

4) Find particular solutions, valid in $R \setminus Z$, to the following differential equations, where $f^{(r)}(x) = D^r f(x)$ and [a] is the least integer not more than a.

(i)
$$f^{(f^{(x)}(x))} = 1$$
. (ii) $f^{(f^{(x)}(x)(x))} = 1$

5) Mr C.T. Commuter, on his way to the underground station in the morning, has to cross Drowntwash Common. This involves crossing a railway bridge, walking over the common, crossing a main road and going down a side road. The traffic distribution is such that the probability of waiting t minutes to cross the main road is

$$p'$$
 waiting less than t minutes) = $(t + T)/2T$

He always malks at the same speed and can cross the common directly in a minutes and then walk to the side road in b minutes.

If he chooses his routes to and from the station as sensibly as possible, does he take longer to cross the common on his way to or from the station? Also work out how long he takes to cross the common on the way to the station.



Common

$$\div\div\div\bigg]\bigg[\div\div\div\div\div\div\div\div\div\div\div\div$$
 Railway

- 6) This problem is the subject of 'The Cups Problem' on p.24. Attorbed
- 7) Solve these simultaneous equations in integers X,Y,U,V between 1 and 50.

$$x^{2} + 5y^{2} = u^{2}$$

 $x^{2} - 5y^{2} = v^{2}$

- 8) Find all finite commutative groups such that the product of all the elements of the group is not the identity.
- 9) Our hero finds himself surrounded by four baddies, at the corners of a square with him at the centre. A wry smile plays across his lips as he assesses the situation; he knows that all four can run at the same speed, which, owing to their not having spent three hours a day training, is just three-cuarters of his. But all of them, like him, have infinitely fast reactions.

Will our rugged hero escape? If so, how and why?

10) G is a finite abelian multiplicative group generated by \mathbf{x}_1 , \mathbf{x}_2 which commute. Entries in this cross-group are elements of G, written with a power of \mathbf{x}_1 or \mathbf{x}_2 only in each cell.

Across. i. $\sqrt{x_1}$ 3. $(x_1x_2)^n$ where n is the smallest positive integer such that the problem has a unique solution.

Down. 1. $\sqrt[5]{x_1}$ 2. $\sqrt[3]{x_1}$



11) In this alphametric, each letter represents a digit in the base of ten. There are no leading zeroes. What is the value of THISVISIT?

THEINVARIANTS

- + ARCHIMEDEANS
- = PROBLEMSDRIVE
- 12) "We can find the probability that a number is prime by two different methods.
- (i) The number of primes < n is denoted by P'(n). It is well known that $P'(n) \sim n/\log n$ as $n \to \infty$. Hence p'(n) = p'(n)
- 'ii) Obviously $p'n \ \text{is prime}) = p(2 \ \text{fn \& ...}) \ \text{where the ellipsis indicates all primes less than n. Since the events}$

2 | n, 3 | n, ... are independent,

 $p(n \text{ is prime}) = \prod p(p \nmid n) = \prod (1-1/p).$

It can be shown (honestly!) that

$$\prod_{p \le n} (1-1/p) \sim 2\exp(-g)/\log n$$

as n \rightarrow ∞ , where g is Euler's constant, g = 1.124... . Thus

p'n is prime) $\sim 1.124/\log n$

This is a contradiction, and the end of mathematics."

What is "rong with the above argument?

13) Two cicuits of model car track are laid out, crossing at several points. Circuit A is of length 60, circuit B of length 35 and the crossing points relative to the cars starting point are at distances:

A 7 15 18 34 48 57 B 9 14 17 19 27 63

9 14 1/ 19 2/ 05

Do the cars crash, and if so where?

14) What is the next number in the following sequences?

(a) 3,3,5,4,4,3,5,5,4 ? (b) 1,2,720,? (c) 2,1,13,19 97,211,? (d) 6,8,5,8,4,0,7,3,4,?

Exchanges/

The following periodicals were exchanged with Eureka this year. They have been placed in the Scientific Periodicals Library.

Gezeta Mathematica seria A,B; Journel of the Mathematical Association of Ghan; Scientia Sinica; Studia Scientiarum Mathematicarum Hungarica; Analele Stiintifice ale Universitatii "Al I. Cuza"; Annales Universitatio Scientarium Budapestensis de Rolando Eotvos Nominatae; Nordisk Matematiski Tidiskrift; Glasnik Matematicki; Revue Roumanie de Mathematiques Pures et Appliquees; Analele Universitatii din Timisoare; Science et Techniques; Abstracts of Bulgarian scientific Literature.

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affirmative using a very ingenious argument as our justification.

Let a be the rule of kal vahomer, and b the rule () I- q

when q is an axiom. This merely asserts that an axiom is also a theorem. Let p(x) be "a statement provable by use of b can be used as a premiss in an inference by x.". Then p(a) is true, and p'b) is true. No- let q'y) be "a statement provable by use of y can be used as a premiss in an inference by a.". Then p'a) = q'b) and so is provable (indeed, an axiom, as far as we are concerned). No- q(a) is certainly undecidable, and p > p gives w(a,b).

Thus by kal vahomer in this system, we derive n(a), that

is, the extended use of kal vahomer.

Please note that this is by no means a formal proof. It is merely arguing that if kel vahomer applies in the formal language, it is reasonable to admit it in the informal metalanguage: but this then velidates the extended form in T.

We have thus achieved a resonable extension of the predicate caculus. What is so remarkable is that this was originally argued in a system guite distinct from modern logic, in which, some 500 years ago, the importance of a metalanguage, and undecidable (or theku) statements was realised. Perhaps our modern mathematics is not quite so modern after all!

References.

A fuller statement and bibliography is in

P. Longworth Confrontations with Judaism (1966).pp. 171-196 There is a detailed account in

H. Guggenheimer "Uber ein bemerkenswertes logisches System in der Antike", Methods (1951)

ANSWERS (CONT.)

11) 769369397. D.M.B.C are arbitrary, except that B = C, and the rest pre

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- 12) The second argument is faulty. The events '2 | n', ... are independent in pairs, but not in threes, for if n is divisible by the two largest primes less than its square root, it will never be divisible by the third largest such prime.
- 13) Yes, after 334 units.
- 14) (a) 3. They are the number of letters in 'one','tho',... (b) 620 448 401 733 239 439 360 000 = 24! = 4!! (c) 793. They are $3^n + (-2)^n$. (d) 6. They are the decimal digits of 10π .

Answers

by Colin Vout and Martin Brown

- 1) (a) A knot. (b) No. (c) Yes. (d) No.
- 2) (i) $d(x) = (a_1 + 1)(a_2)(a_n + 1)$

(ii)
$$s'(x) = (1 - (-1)^{d(x)})/2$$

(iii)
$$N(x) = \sum_{0}^{x} S(y)S'x-y = [\sqrt{x}]-(x-3)/4 +$$

$$\sum_{0}^{x_{y}} (-1^{d(x)-d(x-y)})/4$$

3) There are one frame with four and two with six sticks.







5) He takes longer going to work. He should aim for the point x minutes from the bridge, "here x minmizes

$$\sqrt{(u^2 + v^2) + (b + T - x)^2/4T}$$

- 6) See page 24.
- 7) X = 41, Y = 12, U = 49, V = 31
- 8) The groups must have just one element of order 2, so the only possible groups are $C_{n-1} \times A$ there n and |A| are odd and A is abelian .
- 9) Our hero does escape, He should head directly for one bad guy. When the angle subtended at him by this and every other bad guy is at least 2prcsin 3'4, which is bound to happen, he belts off along an angle bisector and lives happily ever after, composing problems for Eureka,
- 10) N = 8. The solution is

$$\begin{bmatrix} 1 & X_1^3 & {}^2 X_1^0 \\ {}^3 & X_2^2 & X_1^2 \end{bmatrix}$$

The Cups Problem

by M. Brown

Question.

If M cups originally upright are inverted N at a time, so that eventually some that have been turned over will be turned upright again.

(a) when is it not possible to have all M cups upside down at some stage?

(b) when it is possible, estimate the minimum number of turns required to carry out the procedure.

Solution.

Let the number of cups unturned at the r'th turn be M_{Γ} , $r=0,12,\ldots$. Then the number unturned is $M-M_{\Gamma}$. Let x_{Γ} be the number of inverted cups which are turned upright at the r'th move, then $N-x_{\Gamma}$ of the unturned cups are turned upside down, hence

$$\mathbf{M}_{r+1} = \mathbf{M}_r - \mathbf{N} + 2\mathbf{x}_r$$

$$\Rightarrow \qquad \mathbf{M}_{r} = \mathbf{M} - \mathbf{r}\mathbf{N} + 2\sum_{i}^{r} \mathbf{x}_{q} : \mathbf{r} > 0 \qquad 2$$

Hence the problem is soluble if there is an r such that

$$0 = M - RN + 2 \sum x_0$$

Hence there is no solution if M is odd and N even, and we shall prove that in all other cases there is a solution. This answers (a).

At each turn "e have

$$0 < \mathbf{x}_r < \mathbf{N}$$

$$M_r - N + x_{r+1} > 0$$

$$M - M_r - x_{r+1} > 0$$

$$= > (\mathbf{r} - 1)\mathbf{N} \ge \mathbf{x}_r + 2\sum_{i=1}^{r-1} \mathbf{x}_q \ge r\mathbf{N} - \mathbf{M}$$

$$\mathbf{x}_1 = 0$$
 , $\mathbf{x}_R = 0$

The problem is to find a minimum R for (3),(5),(8),(9) to hold for each $r \le R$. Let M = zN + h, $z,h \in Z$ and $0 \le h < N$.

Clearly if N \mid M then the minimal number of turns is M $\!\!\!/N$; thus we can suppose that h > 0.

En.3 yields

$$0 = (\mathbf{z} - \mathbf{R})\mathbf{N} + \mathbf{h} + 2\sum_{i=1}^{R} \mathbf{x}_{q}$$
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and hence we have

$$R > z = \lceil M/N \rceil$$

'.here [a] denotes the greatest integer < a).

If z is sufficiently large, then it is clear that for a solution to exist, we choose x, at each move to cancel h.

Suppose now that N and h are even; thus N and M are even. If we choose $\mathbf{x}_2=(N-h)/2$ then (8) holds for $\mathbf{r}=2$ if $\mathbf{z}>1$ but if we choose $\mathbf{x}_3=\mathbf{x}_4=\ldots=0$, we see (8) holds for all $\mathbf{r}<\mathbf{z}=2$ and for some R, (3),(5),(8),(9) hold; in fact we choose $R=\mathbf{z}+1$. This is obviously the minimum number of moves, from (11).

Suppose now that N is odd, M is even and z>1. Then if z is even, h is even and we choose $x_z=N-h/2$, with $x_3=x_4=\ldots=0$. This is a solution of (3),(5),(8),(9) for R=z+2. From (4),'(11) we have that this is the minmum number of moves.

If now z is odd, then h is odd and we can choose as before $x_2 = n - h/2$, $x_3 = x_4 = ... = 0$ and get a solution in z + 1 moves, which is manifestly minimal.

The remaining case for z > 1 is n odd and m odd.

If z is even then h is odd and there is a solution in z + 1 moves with $x_z = (N - h)/2$, etc.

If z is odd then h is even and there is a solution in z + 2 moves with x_z = N = h/2, etc.

Hence for z > 1, "e collect these in the formula

$$R = [M/N] + {}^{\prime}1 + (-1)^{N})/2 + (1 - {}^{\prime}-1)^{N}/3 + (-1)^{M+DM/N})/4 + (2)$$

We now consider z = 1.

Lemma.

If M and N are both even or both odd and [M'N] = 1, N f M, then there is a solution in 3 moves and any solution in 3 moves ith [M'N] = 1 must have M = N mod 2), and for no such M,N is there a minimal solution in 2r + 1 moves, r > 1. Proof.

If there is a solution in 3 moves,

$$0 = M - 3N + 2(x_1 + x_2 + x_3)$$

but $\mathbf{x}_1 = \mathbf{x}_2 = 0$, so

$$\mathbf{x_2} = (3\mathbf{N} - \mathbf{M})/2$$

=> M,N are both even or both odd. Conversely, if M = N $\pmod{2}$ and $\binom{M}{N}$ = 1, then (3N - M)/2 is an integer and

$$0 \le (3N - M)/2 \le N$$

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so we take $\mathbf{x}_2=(3N-M)/2$ and we have a 3-move solution. Suppose there were a minimal solution in 2r+1 moves, r>1, for some M,N with [M/N]=1. Then, by (4), M=N (mod 2) and so there is a 3-move solution, which is a contradiction.

Q.E.D.

We may now suppose N odd, M even. Suppose that there is a solution in an even number of moves, n for M.N. Consider the complementary position with M - N cups turned over on each move, and M cups altogether. Then it is clear that the n-move solution for M N also furnishes an n-move solution for M,M N and after n moves in the M,M-N problem the cups will all be the same way up.

Let m be a minimal solution for M,M-N and suppose there is a minimal solution for M,N in n < m moves. If n is odd then from the lemma , n = 3 so N = M (mod 2), a contradiction so n is even and so there is a solution for M,M - N in n moves, another contradiction. Obviously m < n since m is even from (4). So the minimal number of moves for M,M-N equals the minimal number for M,N if [M'N; = 1, when M is even and N odd.

3 Problems

by Denis Thicket

- (1) Define sequences s (i 0,1,2...) of 0's and 1's as follows: $\mathbf{s}_0 = 0$, $\mathbf{s}_1 = 1$, and for i > 2. $\mathbf{s}_i = \mathbf{s}_{i-1}$ followed by \mathbf{s}_{i-2} . Let \mathbf{s}_{∞} be the sequence 10110101... Of which each \mathbf{s}_i ($i \neq 0$) is an initial segment. Give a rule for finding the n'th term of \mathbf{s}_{∞} without writing the series out, and generalize.
- (2) Prove by elementary means: if q is the least prime factor of the order of a finite group G then any subgroup of index q in G is normal.
- (\S) Prove or disprove the anti-Fermat conjecture: if N,a,b,c are positive integers such that

$$a^{N^{-1}} - b^{N^{-1}} = c^{N^{-1}}$$

then there are positive integers t,x,y such that

$$a = tx^{N}, b = ty^{N}, c = t(x + y)^{N}.$$

How to toss a coin

by A. Smith

Before you answer that, perhaps I should point out that the problem is, as it stands, rather meaningless. Mathematicians at Oxford have not been idle recently, however, and we have now discovered a context in which the question becomes a perfectly sensible one.

The story goes as follows. An undergraduate possesses a coin, which has probability p of showing heads when tossed. He invites a friend to guess the value of p, and to make the game more interesting, he suggests a gamble along the following lines. If the friend guesses the value g, then he is to receive a sum equal to $f(c - k(p - g)^2)$ if the latter is positive, but if negative he pays the owner of the coin a corresponding amount. (Assume that the values of c and k are such as to make the gamble interesting). The friend hesitates for a moment and then asks whether he is allowed to toss the coin a fer time "just to get the feel of it". (You see, the friend knows his laws of large numbers and reasons that, if he can toss it enough times, the proportion of heads that will serve as his guess g is very likely to be close to p the smaller $(g - p)^2$, the bigger his minnings). The owner replies that he can chhoose to make any fixed number of tosses, but it ill cost him £c per toss, plus £d (for luck) every time a head appears.

Question: how many tosses should the friend choose to make? Should he in fact play the game at all?

We first notice that if he were to make n tosses, r of which turned out to be heads, and then were to guess g for p, then his loss, with respect to guessing g without any tosses would be given by $k^\prime p - g)^2 \perp cn + dr$. But what should g be? Cleerly it should depend on his beliefs about p posterior to performing the tosses: these in turn, depend on his beliefs prior to performing the tosses. Let us suppose the latter to be represented by $\pi^\prime p) = L p^{a-l^\prime} + p)^{b-l}$, where

$$1^{\prime}L = \int_{0}^{1} p^{a-1/(a-p)^{b-1}} dp = \Gamma(a)\Gamma(b)/\Gamma(a+b)$$

and

$$\int_{I} \pi \langle p \rangle \ dp \ / \!\! \int_{\widetilde{I}} \pi \, \langle p \rangle \ dp$$

is the odds the friend vould give on p belonging to the interval I rather then its complement Υ it is through the latter kind of consideration that one chooses the values a and b).

Bayes theorem nom provides us with the required posterior