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Finally, in a forthcoming paper we plan to discuss the particular cases $k = 3$ and $k = 4$.

m/j	1	2	3	4	5	6	7	8	9
1	5	14	30	55	91	140	204	285	385
2	13	29	54	90	139	203	284	384	505
3	25	50	86	135	199	280	380	501	645
4	41	77	126	190	271	371	492	636	705
5	61	110	174	255	355	476	620	689	885
6	85	149	230	330	451	595	664	860	1085
7	113	194	294	415	559	628	824	1049	1305
8	145	245	366	510	679	875	1100	1356	1645
9	181	302	446	615	811	1036	1292	1581	1905

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FIG. 1.

Reference

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AN INTERESTING CONTINUED FRACTION

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I. Introduction. Consider the following continued fraction

$$\alpha = 1 + \frac{b-2}{2\,-}\;\frac{b+2}{b+1\,-}\;\frac{1}{b\,-}\;\frac{1}{b\,-}\;\frac{1}{b\,-}\;\cdots \qquad (b \geqq 2) \tag{1}$$

This continued fraction and its convergents have many unusual properties. In fact, the numerators and denominators of the convergents to (1) form many sequences that occur in number-theoretic problems.

II. Value of α. Since the related continued fraction

$$\beta = \frac{1}{b\,-}\;\frac{1}{b\,-}\;\frac{1}{b\,-}\;\cdots \tag{2}$$

is easily shown to be equal to $\frac{1}{2}(b - \sqrt{b^2 - 4})$, it readily follows that $\alpha = \frac{1}{2}(b + \sqrt{b^2 - 4})$. It is also obvious that α is the conjugate of β and that $\alpha = 1/\beta$. α and β are the roots of the quadratic $x^2 - bx + 1 = 0$.

The numbers α and β have the property that each number plus its reciprocal equals b:

$$\alpha + 1/\alpha = \beta + 1/\beta = b.$$

The *simple* continued fraction expansions (as contrasted with (1) and (2), which are irregular) of both α and β are interesting:

$$\alpha = b - 1 \, \frac{1}{+1} \, \frac{1}{+b-2} \, \frac{1}{+1} \, \frac{1}{+b-2+} \cdots$$

$$\beta = \frac{1}{b-1} \, \frac{1}{+1} \, \frac{1}{+b-2} \, \frac{1}{+1} \, \frac{1}{+b-2+} \cdots$$

In certain cases, $\sqrt{\alpha}$ and $\sqrt{\beta}$ are also quadratic irrationals and not quartic (biquadratic) irrationals. For we have

$$\sqrt{\alpha} = (\sqrt{b} + \sqrt{b^2-4})/\sqrt{2} = \tfrac{1}{2}(\sqrt{b+2} + \sqrt{b-2}).$$

Now if $b = x^2 + 2$, then

$$\sqrt{\alpha} = \tfrac{1}{2}(x + \sqrt{x^2+4}) = x \, \frac{1}{+x} \, \frac{1}{+x} \, \frac{1}{+x+} \cdots$$

If $b = x^2 - 2$, then

$$\sqrt{\alpha} = \tfrac{1}{2}(x + \sqrt{x^2-4}) = x - 1 \, \frac{1}{+1} \, \frac{1}{+x-2} \, \frac{1}{+1} \, \frac{1}{+x-2+} \cdots$$

There are similar expansions for $\sqrt{\beta}$. We have

$$\sqrt{\beta} = (\sqrt{b} - \sqrt{b^2-4})/\sqrt{2} = \tfrac{1}{2}(\sqrt{b+2} - \sqrt{b-2}).$$

If $b = x^2 + 2$, then

$$\sqrt{\beta} = \tfrac{1}{2}(\sqrt{x^2+4} - x) = \frac{1}{x} \, \frac{1}{+x} \, \frac{1}{+x} \, \frac{1}{+x+} \cdots$$

If $b = x^2 - 2$, then

$$\sqrt{\beta} = \tfrac{1}{2}(x - \sqrt{x^2-4}) = \frac{1}{x-1} \, \frac{1}{+1} \, \frac{1}{+x-2} \, \frac{1}{+1} \, \frac{1}{+x-2+} \cdots$$

III. Convergents to (1). The first few convergents, p_n/q_n, to (1) for $b = 2, 3, 4, 5, 6$ and $n = 1, 2, 3, 4, 5, 6, 7, 8, 9$ are given in Table I.

By the rule for determining the convergents to a continued fraction, we have

$$p_1/q_1 = 1/1, \quad p_2/q_2 = b/2, \quad p_3/q_3 = (b^2-2)/b.$$

Also, for $n > 1$, we have $p_n = q_{n+1}$.

TABLE I

Convergents to (1)

$b \backslash n$	1	2	3	4	5	6	7	8	9
2	$\frac{1}{1}$	$\frac{2}{2}$	$\frac{2}{2}$	$\frac{2}{2}$	$\frac{2}{2}$	$\frac{2}{2}$	$\frac{2}{2}$	$\frac{2}{2}$	$\frac{2}{2}$
3	$\frac{1}{1}$	$\frac{3}{2}$	$\frac{7}{3}$	$\frac{18}{7}$	$\frac{47}{18}$	$\frac{123}{47}$	$\frac{322}{123}$	$\frac{843}{322}$	$\frac{2207}{843}$
4	$\frac{1}{1}$	$\frac{4}{2}$	$\frac{14}{4}$	$\frac{52}{14}$	$\frac{194}{52}$	$\frac{724}{194}$	$\frac{2702}{724}$	$\frac{10084}{2702}$	$\frac{37634}{10084}$
5	$\frac{1}{1}$	$\frac{5}{2}$	$\frac{23}{5}$	$\frac{110}{23}$	$\frac{527}{110}$	$\frac{2525}{527}$	$\frac{12098}{2525}$	$\frac{57965}{12098}$	$\frac{277727}{57965}$
6	$\frac{1}{1}$	$\frac{6}{2}$	$\frac{34}{6}$	$\frac{198}{34}$	$\frac{1154}{198}$	$\frac{6726}{1154}$	$\frac{39202}{6726}$	$\frac{228486}{39202}$	$\frac{1331714}{228486}$

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Now consider the Fibonacci-like sequence defined by the second order recurrence

$$a_n = ba_{n-1} - a_{n-2}, \qquad a_0 = 2, \qquad a_1 = b.$$

By the theory of difference equations, it can be shown that

$$a_n = [\tfrac{1}{2}(b + \sqrt{b^2-4})]^n - [\tfrac{1}{2}(b - \sqrt{b^2-4})]^n. \tag{3}$$

But $\alpha = \frac{1}{2}(b + \sqrt{b^2-4})$, $\beta = \frac{1}{2}(b - \sqrt{b^2-4})$. Therefore, $a_n = \alpha^n + \beta^n$. By induction, it is easily demonstrated that $a_n = p_{n-1} = q_{n+2}$. From equation (3) it also easily follows that $a_{2n} = a_n^2 - 2$.

IV. The case $b = 3$. If $b = 3$, then $\alpha = \frac{1}{2}(3 + \sqrt{5}) = \phi + 1$, where ϕ is phi, the golden ratio [1], and $\beta = \frac{1}{2}(3 - \sqrt{5}) = 2 - \phi$.

The sequence of numerators, p_n, to the continued fraction in equation (1) is

$$3, 7, 18, 47, 123, 322, 843, 2207, \cdots$$

In fact, for $n > 1$, $p_n = L_{2n-2}$, where L_n is the Lucas sequence defined by $L_0 = 2$, $L_1 = 1$, $L_n = L_{n-1} + L_{n-2}$ [2].

Also, $p_{2^n+1} = r_n$ is another one of the sequences studied by Lucas [3], defined by $r_0 = 3$, $r_{n+1} = r_n^2 - 2$. This sequence

1566

$$3, 7, 47, 2207, 4870847, \cdots$$

was employed by Lucas to test the primality of Mersenne numbers of the form $2^{4m+3} - 1$, where $4m + 3$ is prime.

Sierpinski [4] noted that

$$\beta = \tfrac{1}{2}(3-\sqrt{5}) = 2-\phi = \frac{1}{r_0} + \frac{1}{r_0 r_1} + \frac{1}{r_0 r_1 r_2} + \frac{1}{r_0 r_1 r_2 r_3} + \cdots$$

V. The case $b = 4$. If $b = 4$, then $\alpha = 2+\sqrt{3}$, $\beta = 2-\sqrt{3}$, and p_n is the sequence

$$4,\ 14,\ 52,\ 194,\ 724,\ 2702,\ 10084,\ 37634,\ \cdots$$

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$p_{2^n-1} = s_n$ is another sequence discussed by Lucas, defined by $s_0 = 4$, $s_{n+1} = s_n^2 - 2$. Lucas employed this sequence to test the primality of Mersenne numbers [5]. Lehmer [6] improved the test to the following form:

If n is an odd prime, then $2^n - 1$ is prime if and only if it evenly divides s_{n-1}. The sequence s_n is

$$4,\ 14,\ 194,\ 37634,\ 1416317954,\ \cdots$$

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VI. The case $b = 6$. If $b = 6$, then $\alpha = 3+2\sqrt{2}$, $\beta = 3-2\sqrt{2}$, and p_n is the sequence

$$6,\ 34,\ 198,\ 1154,\ 6726,\ 39202,\ \cdots$$

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This sequence is involved in the determination of whether or not the product of three consecutive triangular numbers, $T_{n-1}T_nT_{n+1}$, is a square. In fact, $T_{n-1}T_nT_{n-1}$ is a square if $n = (3p_k - 2)/4$. See Beiler [7].

$P_{2^n-1} = v_n$ is still another sequence discussed by Lucas [8]. The sequence v_n is as follows:

$$6,\ 34,\ 1154,\ 1331714,\ \cdots$$

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where $v_0 = 6$, $v_{n+1} = v_n^2 - 2$. This sequence was employed by Lucas to test the primality of Fermat numbers $2^{2^n} + 1$.

VII. The case $b = \sqrt{5}$. This case is rather unusual because b is not an integer, so none of the convergents except p_1/q_1 represent rational numbers. We have $\alpha = \tfrac{1}{2}(1+\sqrt{5}) = \phi$ and $\beta = \tfrac{1}{2}(\sqrt{5}-1) = \phi - 1$. The first 9 convergents to (1) with $b = \sqrt{5}$ are given in Table II.

TABLE II

p_n/q_n for $b = \sqrt{5}$

n	1	2	3	4	5	6	7	8	9
p_n/q_n	$\frac{1}{1}$	$\frac{\sqrt{5}}{2}$	$\frac{3}{\sqrt{5}}$	$\frac{2\sqrt{5}}{3}$	$\frac{7}{2\sqrt{5}}$	$\frac{5\sqrt{5}}{7}$	$\frac{18}{5\sqrt{5}}$	$\frac{13\sqrt{5}}{18}$	$\frac{47}{13\sqrt{5}}$

From equation (3) it is easy to show that $p_{2n}/\sqrt{5} = F_{2n-1}$, where F_n is the famous Fibonacci sequence defined by $F_0 = 0$, $F_1 = 1$, $F_n = F_{n-1} + F_{n-2}$. The first few terms of the Fibonacci sequence are

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, \cdots$$

From equation (3) it also can be shown that $p_{2n+1} = L_{2n}$, where L_n is the Lucas sequence discussed in part IV.

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QUARTIC EQUATIONS AND TETRAHEDRAL SYMMETRIES

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1. Introduction. In Section 2, we give a short derivation of formulas for the roots of a quartic equation. A closely related representation of the symmetric group S_4 by matrices of size 3×3 is presented in Section 3. Geometric interpretations follow in Section 6.

Throughout, let F be a field in which $1 + 1 \neq 0$. For us, the matrix

$$H = \tfrac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

is basic. It has $H^2 = I$ and $H^{-1} = H$.