In sequence A005186, a_n is the number of positive integers which take n steps to reach 1 in the Collatz iteration. A comment in A005186 says that, according to David W. Wilson, a heuristic argument indicates that a_{n+1}/a_n converges to the largest eigenvalue of the matrix

$$M := \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{3} & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & \frac{1}{3} & 0 \end{pmatrix} \quad , \quad \text{which is} \quad \frac{1}{6}\left(3+\sqrt{21}\right).$$

This claim is supported by Hugo Pfoertner's visualization of a_{n+1}/a_n, see A005186.png.

The heuristic argument is not supplied. It might work like this:

Let A_n be the set of positive integers that take n steps to reach 1. For $0 \le i \le 5$ set

$$B_{n,i} := \{x \in A_n : x \equiv i \pmod 6\} \quad \text{and} \quad b_{n,i} := |B_{n,i}|\,,$$

We have $A_{n+1} = C_n \uplus D_n$ with $C_n := \{2k : k \in A_n\}$ and

$$\begin{aligned} D_n \quad &:= \quad \left\{\frac{k-1}{3} : k \in A_n, k > 4, k \equiv 1 \pmod 3, \frac{k-1}{3} \equiv 1 \pmod 2\right\} \\ &= \quad \left\{\frac{k-1}{3} : k \in A_n, k > 4, k \equiv 4 \pmod 6\right\}. \end{aligned}$$

Let $d_{n,i} := |\{x \in D_n : x \equiv i \pmod 6\}|$. Then we have:

$$\begin{aligned} b_{n+1,0} &= b_{n,0} + b_{n,3} + d_{n,0} \quad , & b_{n+1,1} &= d_{n,1} \\ b_{n+1,2} &= b_{n,1} + b_{n,4} + d_{n,2} \quad , & b_{n+1,3} &= d_{n,3} \\ b_{n+1,4} &= b_{n,2} + b_{n,5} + d_{n,4} \quad , & b_{n+1,5} &= d_{n,5} \end{aligned}$$

The condition $(k-1)/3 \equiv 1 \pmod 2$ gives $d_{n,0} = d_{n,2} = d_{n,4} = 0$. Because of the condition $k \equiv 4 \pmod 6$, only elements of $B_{n,4}$ can contribute to D_n. The heuristic is to assume that, for large n, the values $(k-1)/3$ for $k \in B_{n,4}$ are distributed evenly in the odd residue classes modulo 6, so $d_{n,1}, d_{n,3}$, and $d_{n,5}$ each is approximately equal to $b_{n,4}/3$. Then the vector $(b_{n+1,0}, ..., b_{n+1,5})$ is the product of a 6×6 matrix M_n with the vector $(b_{n,0}, ..., b_{n,5})$, where $M_n \to M$ for $n \to \infty$. The (normalized) vectors converge to an eigenvector of M for its largest eigenvalue.

Looking at residue classes modulo 2 gives a simpler argumentation and a smaller matrix: Let e_n be the number of even and f_n the number of odd elements of A_n. Then $e_{n+1} = |C_n| = e_n + f_n$. Again assuming that about every third even element of A_n is congruent to 4 modulo 6 yields that $f_{n+1} = |D_n|$ is approximately equal to $e_n/3$. Hence, the evolution of (e_n, f_n) is approximately given by the matrix

$$\begin{pmatrix} 1 & 1 \\ \frac{1}{3} & 0 \end{pmatrix},$$

which has the same largest eigenvalue $\frac{1}{6}\left(3+\sqrt{21}\right)$.

This latter approach also allows to deduce the asymptotic ratio of even and odd elements of A_n: Using $e_{n+1} = e_n + f_n$, and taking $\lim f_{n+1}/e_n = 1/3$ for granted, it follows easily that $\lim e_n/f_n = \frac{1}{2}\left(3+\sqrt{21}\right)$, see A090458.

Markus Sigg, 2020-11-27