Some simple continued fraction expansions associated with A005169, A111317 and A143951.

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We find the simple continued fraction expansions of the real numbers $F\left(\frac{1}{N}\right)$, N a nonzero integer, where $F(x)$ denotes the ordinary generating function of one of the sequences A005169, A111317 or A143951. It turns out that these continued fraction expansions are non-terminating showing the numbers $F\left(\frac{1}{N}\right)$ to be irrational.

Preliminaries on continued fractions

We adopt the standard compact notation

$$a_0 + \frac{a_1}{b_1 +} \frac{a_2}{b_2 +} \frac{a_3}{b_3 +} \cdots$$

to denote the general continued fraction

$$a_0 + \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \cfrac{a_3}{b_3 + \cdots}}}. \tag{1}$$

We refer to the terms $a_n, n \geq 1$, in (1) as the *partial numerators* and the terms b_n as the *partial denominators* of the continued fraction. A *simple continued fraction* is a continued fraction in which a_0 is an integer, all the partial numerators are equal to 1 and each partial denominator is a positive integer. We recall (see for example [2, Theorem 14]) that every positive irrational real number has a unique expansion as a simple continued fraction (with an infinite number of terms). Rational numbers have finite simple continued fraction expansions.
Given a sequence λ_n of non-zero complex numbers, the continued fraction

$$a_0 + \cfrac{\lambda_1 a_1}{\lambda_1 b_1 + \cfrac{\lambda_1 \lambda_2 a_2}{\lambda_2 b_2 + \cfrac{\lambda_2 \lambda_3 a_3}{\lambda_3 b_3 + \cdots}}} \quad (\lambda_n \neq 0) \tag{2}$$

is said to be obtained from (1) by means of an *equivalence transformation.* The continued fractions (1) and (2) are equivalent in the sense that the n-th convergents of both fractions have the same value for all n [3, p.19]. If in (1) the partial numerators a_n are all nonzero then we can clearly choose complex numbers $\lambda_1, \lambda_2, \lambda_3$,... so that

$$1 = \lambda_1 a_1 = \lambda_1 \lambda_2 a_2 = \lambda_2 \lambda_3 a_3 = \dots\ .$$

By this means we can arrange that the partial numerators in the equivalent continued fraction (2) are all equal to 1. The following two lemmas are useful for converting certain types of continued fractions into ones with all partial numerators equal to $+1$.

Lemma 1. *If* $a_1, a_2, \ldots, a_n$ *is a sequence of complex numbers then*

$$\begin{aligned}\frac{1}{a_1-}\,\frac{1}{a_1+}\,\frac{1}{a_2-}\,\frac{1}{a_2+}\cdots\frac{1}{+a_n-}\,\frac{1}{a_n} \quad = \quad & \frac{1}{a_1-1+}\,\frac{1}{1+}\,\frac{1}{a_1-1+}\,\frac{1}{a_2-1+}\,\frac{1}{1+}\,\frac{1}{a_2-1+} \\ & \frac{1}{+a_3-1+}\,\frac{1}{1+}\cdots\frac{1}{+a_n-1+}\,\frac{1}{1+}\,\frac{1}{a_n-1}.\end{aligned} \tag{3}$$

Proof
A proof by induction is immediate once we note the algebraic identity

$$\frac{1}{a_1-}\,\frac{1}{a_1+X}=\frac{1}{a_1-1+}\,\frac{1}{1+}\,\frac{1}{a_1-1+X}.$$

■

In order to prove Lemma 2 we will need the following preliminary result.

Proposition 1. *If* $a_1, a_2, \ldots, a_n$ *is a sequence of complex numbers then*

$$1+\frac{1}{a_1-1+}\,\frac{1}{a_2+}\cdots\frac{1}{+a_n}=\frac{1}{1-}\,\frac{1}{a_1+}\,\frac{1}{a_2+}\cdots\frac{1}{+a_n}. \tag{4}$$

Proof. By induction on n. The result is easily verified for $n=1$. Assume that (4) is true for a fixed integer $n \geq 1$. Then by induction

$$\begin{aligned}1+\frac{1}{a_1-1+}\,\frac{1}{a_2+}\cdots\frac{1}{+a_{n+1}} &= 1+\frac{1}{a_1-1+}\,\frac{1}{a_2+}\cdots\frac{1}{+a_{n-1}+}\,\frac{1}{\left(a_n+\frac{1}{a_{n+1}}\right)} \\ &= \frac{1}{1-}\,\frac{1}{a_1+}\,\frac{1}{a_2+}\cdots\frac{1}{+a_{n-1}+}\,\frac{1}{\left(a_n+\frac{1}{a_{n+1}}\right)} \\ &= \frac{1}{1-}\,\frac{1}{a_1+}\,\frac{1}{a_2+}\cdots\frac{1}{+a_{n+1}}\end{aligned}$$

and the induction goes through. ■

Lemma 2. *If* $a_1, a_2, \ldots, a_n$ *is a sequence of complex numbers then*

$$\frac{1}{1-}\,\frac{1}{a_1-}\,\frac{1}{a_2-}\cdots\frac{1}{-a_n}=1+\frac{1}{a_1-2+}\,\frac{1}{1+}\,\frac{1}{a_2-2+}\,\frac{1}{1+}\cdots\frac{1}{+a_n-2+}\,\frac{1}{1}. \tag{5}$$

Proof. By induction on n. The result is easily verified for $n=1$. Assume that (5) is true for a fixed integer $n \geq 1$. Let $F(n)$ denote the rhs of (5). Then

$$\begin{aligned} F(n+1) &= 1 + \frac{1}{a_1 - 2 +}\,\frac{1}{1 +}\,\frac{1}{a_2 - 2 +}\,\frac{1}{1 +} \cdots + \frac{1}{a_n - 2 +}\,\frac{1}{1 +}\,\frac{1}{a_{n+1} - 2 +}\,\frac{1}{1} \\ &= 1 + \frac{1}{a_1 - 2 + \frac{1}{A}}, \end{aligned}$$

where

$$\begin{aligned} A &= 1 + \frac{1}{a_2 - 2 +}\,\frac{1}{1 +} \cdots + \frac{1}{a_n - 2 +}\,\frac{1}{1 +}\,\frac{1}{a_{n+1} - 2 +}\,\frac{1}{1} \\ &= \frac{1}{1 -}\,\frac{1}{a_2 -}\,\frac{1}{a_3 -} \cdots - \frac{1}{a_{n+1}} \end{aligned}$$

by the induction hypothesis. Thus

$$\begin{aligned} F(n+1) &= 1 + \cfrac{1}{a_1 - 2 + 1 - \frac{1}{a_2 -}\,\frac{1}{a_3 -} \cdots - \frac{1}{a_{n+1}}} \\ &= 1 + \frac{1}{a_1 - 1 -}\,\frac{1}{a_2 -}\,\frac{1}{a_3 -} \cdots - \frac{1}{a_{n+1}} \\ &= 1 + \frac{1}{a_1 - 1 +}\,\frac{1}{(-a_2) +}\,\frac{1}{a_3 +} \cdots + \frac{1}{((-1)^n a_{n+1})} \quad \text{(equivalence transformation)} \\ &= \frac{1}{1 -}\,\frac{1}{a_1 +}\,\frac{1}{(-a_2) +}\,\frac{1}{a_3 +} \cdots + \frac{1}{((-1)^n a_{n+1})} \quad \text{(by Proposition 1)} \\ &= \frac{1}{1 -}\,\frac{1}{a_1 -}\,\frac{1}{a_2 -} \cdots - \frac{1}{a_{n+1}}, \end{aligned}$$

where, in the final step, we made use of another equivalence transformation. This completes the proof by induction. ■

We now use these lemmas to find the simple continued fraction expansions for numbers of the form $F\left(\frac{1}{N}\right)$, N a nonzero integer. Here $F(x)$ denotes the ogf for one of the sequences A005169, A111317 and A143951. We consider each sequence in turn.

1) A005169

The ogf $F(x)$ for A005169 is given there as the continued fraction

$$F(x) = \frac{1}{1 -}\,\frac{x}{1 -}\,\frac{x^2}{1 -}\,\frac{x^3}{1 -}\,\frac{x^4}{1 -} \cdots \qquad |x| < 1. \tag{6}$$

We shall establish the following pair of simple continued fraction expansions

$$F\left(\frac{1}{N}\right) = 1 + \frac{1}{N - 2 +}\,\frac{1}{1 +}\,\frac{1}{N - 2 +}\,\frac{1}{1 +}\,\frac{1}{N^2 - 2 +}\,\frac{1}{1 +}\,\frac{1}{N^2 - 2 +} \cdots \tag{7}$$

valid for positive integer $N \geq 3$, and

$$F\left(-\frac{1}{N}\right) = \frac{1}{1+}\,\frac{1}{N-1+}\,\frac{1}{1+}\,\frac{1}{N-1+}\,\frac{1}{N^2-1+}\,\frac{1}{1+}\,\frac{1}{N^2-1+}\cdots \tag{8}$$

valid for positive integer $N \geq 2$.

Firstly, from (6) we have

$$F\left(\frac{1}{x}\right) = \frac{1}{1-}\,\frac{\frac{1}{x}}{1-}\,\frac{\frac{1}{x^2}}{1-}\,\frac{\frac{1}{x^3}}{1-}\,\frac{\frac{1}{x^4}}{1-}\cdots \qquad |x| > 1.$$

The using an equivalence transformation to set the partial numerators equal to -1 we arrive at the expansion

$$F\left(\frac{1}{x}\right) = \frac{1}{1-}\,\frac{1}{x-}\,\frac{1}{x-}\,\frac{1}{x^2-}\,\frac{1}{x^2-}\cdots \qquad |x| > 1. \tag{9}$$

An application of Lemma 2 now yields

$$F\left(\frac{1}{x}\right) = 1 + \frac{1}{x-2+}\,\frac{1}{1+}\,\frac{1}{x-2+}\,\frac{1}{1+}\,\frac{1}{x^2-2+}\,\frac{1}{1+}\,\frac{1}{x^2-2+}\cdots \qquad |x| > 1.$$

In particular, if $N \geq 3$ is a positive integer we obtain the continued fraction expansion

$$F\left(\frac{1}{N}\right) = 1 + \frac{1}{N-2+}\,\frac{1}{1+}\,\frac{1}{N-2+}\,\frac{1}{1+}\,\frac{1}{N^2-2+}\,\frac{1}{1+}\,\frac{1}{N^2-2+}\cdots \tag{10}$$

in which every partial numerator equals 1 and each partial denominator is a positive integer. Thus by uniqueness, (10) must be the simple continued fraction expansion of $F\left(\frac{1}{N}\right)$ and the proof of (7) is complete.

Secondly, from (9) we have

$$F\left(-\frac{1}{x}\right) = \frac{1}{1-}\,\frac{1}{(-x)-}\,\frac{1}{(-x)-}\,\frac{1}{x^2-}\,\frac{1}{x^2-}\,\frac{1}{(-x^3)-}\,\frac{1}{(-x^3)-}\cdots.$$

By means of an equivalence transformation we can arrange that the partial numerators, after the first one, are alternately $+1$ and -1. Thus

$$F\left(-\frac{1}{x}\right) = \frac{1}{1+}\,\frac{1}{x-}\,\frac{1}{x+}\,\frac{1}{x^2-}\,\frac{1}{x^2+}\,\frac{1}{x^3-}\,\frac{1}{x^3+}\cdots.$$

It then follows from Lemma 1 that

$$F\left(-\frac{1}{x}\right) = \frac{1}{1+}\,\frac{1}{x-1+}\,\frac{1}{1+}\,\frac{1}{x-1+}\,\frac{1}{x^2-1+}\,\frac{1}{1+}\,\frac{1}{x^2-1+}\,\frac{1}{x^3-1+}\,\frac{1}{1+}\cdots.$$

Thus if $N \geq 2$ is an integer we have obtained the simple continued fraction expansion

$$F\left(-\frac{1}{N}\right) = \frac{1}{1+}\,\frac{1}{N-1+}\,\frac{1}{1+}\,\frac{1}{N-1+}\,\frac{1}{N^2-1+}\,\frac{1}{1+}\,\frac{1}{N^2-1+}\,\frac{1}{N^3-1+}\,\frac{1}{1+}\cdots$$

and the proof of (8) is complete.

2) A111317

The ordinary generating function $F(x)$ for this sequence has the form of an infinite product

$$F(x) = \prod_{n=0}^{\infty}\left\{\frac{1-x^{3n+2}}{1-x^{3n+1}}\right\} \qquad |x|<1. \tag{11}$$

We shall establish the pair of simple continued fraction expansions

$$F\left(\frac{1}{N}\right) = 1 + \frac{1}{N-1+}\,\frac{1}{1+}\,\frac{1}{N^2-1+}\,\frac{1}{1+}\,\frac{1}{N^3-1+}\,\frac{1}{1+}\cdots \tag{12}$$

$$F\left(-\frac{1}{N}\right) = \frac{1}{1+}\,\frac{1}{N-1+}\,\frac{1}{N^2+1+}\,\frac{1}{N^3-1+}\cdots \tag{13}$$

both valid for positive integer $N \geq 2$.

Ramanujan [1, Entry 19, p.49] found the following continued fraction for the function $F(x)$

$$F(x) = \frac{1}{1-}\,\frac{x}{1+x-}\,\frac{x^3}{1+x^2-}\,\frac{x^5}{1+x^3-}\cdots \qquad |x|<1. \tag{14}$$

Using an equivalence transformation we can put this into the form

$$F(x) = \frac{1}{1-}\,\frac{1}{1+\frac{1}{x}-}\,\frac{1}{1+\frac{1}{x^2}-}\,\frac{1}{1+\frac{1}{x^3}-}\cdots \qquad 0<|x|<1.$$

It follows from Lemma 2 that

$$F(x) = 1 + \frac{1}{\frac{1}{x}-1+}\,\frac{1}{1+}\,\frac{1}{\frac{1}{x^2}-1+}\,\frac{1}{1+}\,\frac{1}{\frac{1}{x^3}-1+}\,\frac{1}{1+}\cdots \qquad 0<|x|<1.$$

Hence for integer $N \geq 2$ we get the simple continued fraction expansion

$$F\left(\frac{1}{N}\right) = 1 + \frac{1}{N-1+}\,\frac{1}{1+}\,\frac{1}{N^2-1+}\,\frac{1}{1+}\,\frac{1}{N^3-1+}\,\frac{1}{1+}\cdots.$$

and (12) has been proved.

Similarly, from Ramanujan's continued fraction (14) we have

$$\begin{aligned} F(-x) &= \frac{1}{1+}\,\frac{x}{1-x+}\,\frac{x^3}{1+x^2+}\,\frac{x^5}{1-x^3+}\cdots \qquad |x|<1 \\ &= \frac{1}{1+}\,\frac{1}{\frac{1}{x}-1+}\,\frac{1}{\frac{1}{x^2}+1+}\,\frac{1}{\frac{1}{x^3}-1+}\cdots \qquad 0<|x|<1, \end{aligned}$$

by an equivalence transformation. Thus for integer $N \geq 2$ we get the simple continued fraction expansion

$$F\left(-\frac{1}{N}\right) = \frac{1}{1+}\,\frac{1}{N-1+}\,\frac{1}{N^2+1+}\,\frac{1}{N^3-1+}\cdots$$

completing the proof of (13).

3) A143951

The ogf $F(x)$ for A143951 is given as the continued fraction

$$F(x) = \frac{1}{1-}\,\frac{x}{1-}\,\frac{x^3}{1\ -}\,\frac{x^5}{1\ -}\,\frac{x^7}{1\ -}\cdots \qquad |x|<1. \tag{15}$$

Replacing x with $\frac{1}{x}$ and then carrying out an equivalence transformation yields

$$F\left(\frac{1}{x}\right) = \frac{1}{1-}\,\frac{1}{x-}\,\frac{1}{x^2-}\,\frac{1}{x^3-}\,\frac{1}{x^4-}\cdots \qquad |x|>1.$$

Then using Lemma 2 we find for $N \geq 3$ the simple continued fraction expansion

$$F\left(\frac{1}{N}\right) = 1 + \frac{1}{N-2+}\,\frac{1}{1+}\,\frac{1}{N^2-2+}\,\frac{1}{1+}\,\frac{1}{N^3-2+}\cdots$$

Finally, from (15) we have

$$F(-x) = \frac{1}{1+}\,\frac{x}{1+}\,\frac{x^3}{1\ +}\,\frac{x^5}{1\ +}\,\frac{x^7}{1\ +}\cdots \qquad |x|<1.$$

Replacing x with $\frac{1}{x}$ and then carrying out an equivalence transformation yields the following simple continued fraction expansion

$$F\left(-\frac{1}{N}\right) = \frac{1}{1+}\,\frac{1}{N+}\,\frac{1}{N^2+}\,\frac{1}{N^3+}\cdots$$

valid for positive integer $N \geq 2$.

REFERENCES

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