Scan A5165 etc

R K Guy

Strong law

Add to all the sequences (there are a lot, this is an important paper) finished 5/91 AMM Vol 95 (1988) A5165 - A5171

S728

+ many more

The Strong Law of Small Numbers

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This article is in two parts, the first of which is a do-it-yourself operation, in which I'll show you 35 examples of patterns that *seem* to appear when we look at several small values of n, in various problems whose answers depend on n. The question will be, in each case: do you think that the pattern persists for all n, or do you believe that it is a figment of the smallness of the values of n that are worked out in the examples?

Caution: examples of both kinds appear; they are not all figments!

In the second part I'll give you the answers, insofar as I know them, together with references.

Try keeping a scorecard: for each example, enter your opinion as to whether the observed pattern is known to continue, known not to continue, or not known at all.

This first part contains no information; rather it contains a good deal of disinformation. The first part contains one theorem:

$$\boxed{\text{You can't tell by looking.}}$$

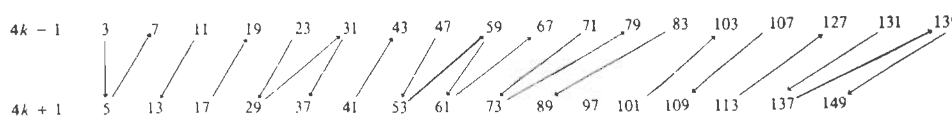
It has wide application, outside mathematics as well as within. It will be proved by intimidation.

Here are some well-known examples to get you started.

Example 1. The numbers $2^{2^0}+1=3$, $2^{2^1}+1=5$, $2^{2^2}+1=17$, $2^{2^3}+1=257$, $2^{2^4}+1=65537$, are primes. A215 and A19434

Example 2. The number 2^n-1 can't be prime unless n is prime, but $2^2-1=3$, $2^3-1=7$, $2^5-1=31$, $2^7-1=127$, are primes. A43 and A668

Example 3. Apart from 2, the oddest prime, all primes are either of shape $4k-1$, or of shape $4k+1$. In any interval $[1, n]$, the former are at least as numerous as the latter ($4k-1$ wins the "prime number race"): A7350, A7351



Example 4. Pick several numbers at random (it suffices just to look at odd ones). Estimate the probability that a number has more divisors of shape $4k-1$, than it does of shape $4k+1$. For example, 21 has two of the first kind (3 & 7) and two of the second (1 & 21), while 25 has all three (1, 5, 25) of the second kind.

Example 5. The five circles of FIG. 1 have $n=1,2,3,4,5$ points on them. These points are in general position, in the sense that no three of the $\binom{n}{2}$ chords joining A127 them are concurrent. Count the numbers of regions into which the chords partition each circle.

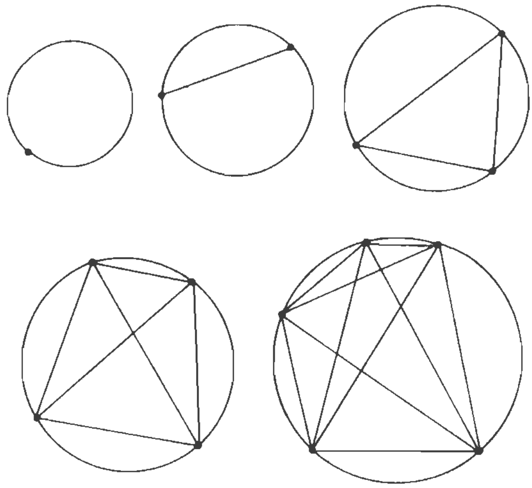


FIG. 1. How many regions in each of these circles?

I've been trying to formulate the **Strong Law of Small Numbers** for many years [**9**]. The best I can do so far is

> There aren't enough small numbers to meet the many demands made of them.

It is the enemy of mathematical discovery. When you notice a mathematical pattern, how do you know it's for real?

> Superficial similarities spawn spurious statements.

> Capricious coincidences cause careless conjectures.

On the other hand, the Strong Law often works the other way:

> Early exceptions eclipse eventual essentials.

> Initial irregularities inhibit incisive intuition.

Here are some misleading facts about small numbers:
Ten per cent of the first hundred numbers are perfect squares.
A quarter of the numbers less than 100 are primes.
Except for 6, all numbers less than 10 are prime powers.
Half the numbers less than 10 are Fibonacci numbers

$$0, 1, 1, 2, 3, 5, 8, \ldots$$

and alternate Fibonacci numbers, $1, 2, 5, \ldots$ are both Bell numbers and Catalan numbers.

Example 6. The numbers 31, 331, 3331, 33331, 333331, 3333331, are each prime. A51200

Example 7. The alternating sums of factorials,

A5165

$$3! - 2! + 1! = 5$$
$$4! - 3! + 2! - 1! = 19$$
$$5! - 4! + 3! - 2! + 1! = 101$$
$$6! - 5! + 4! - 3! + 2! - 1! = 619$$
$$7! - 6! + 5! - 4! + 3! - 2! + 1! = 4421$$
$$8! - 7! + 6! - 5! + 4! - 3! + 2! - 1! = 35899$$

are each prime.

Example 8. In the table A6843

A5728

row 1	1 1	2
row 2	1 2 1	3
row 3	1 3 2 3 1	5
row 4	1 4 3 2 3 4 1	7
row 5	1 5 4 3 5 2 5 3 4 5 1	11
row 6	1 6 5 4 3 5 2 5 3 4 5 6 1	13
row 7	1 7 6 5 4 7 3 5 7 2 7 5 3 7 4 5 6 7 1	19
row 8	1 8 7 6 5 4 7 3 8 5 7 2 7 5 8 3 7 4 5 6 7 8 1	23
row 9	1 9 8 7 6 5 9 4 7 3 8 5 7 9 2 9 7 5 8 3 7 4 9 5 6 7 8 9 1	29

row n is obtained from row $n - 1$ by inserting n between each pair of consecutive numbers which add to n. The number of numbers in each row is shown on the right. Each is prime.

Example 9. Is there a prime of shape $7013 \times 2^n + 1$?

Example 10. Are all the numbers $78557 \times 2^n + 1$ composite?

Example 11. When you use Euclid's method to show that there are unboundedly many primes:

$$2 + 1 = 3$$
$$(2 \times 3) + 1 = 7$$
$$(2 \times 3 \times 5) + 1 = 31$$
$$(2 \times 3 \times 5 \times 7) + 1 = 211$$
$$(2 \times 3 \times 5 \times 7 \times 11) + 1 = 2311$$

you don't always get primes:

$$(2 \times 3 \times 5 \times 7 \times 11 \times 13) + 1 = 30031 = 59 \times 509$$
$$(2 \times 3 \times 5 \times 7 \times 11 \times 13 \times 17) + 1 = 510511 = 19 \times 97 \times 277$$
$$(2 \times 3 \times 5 \times 7 \times 11 \times 13 \times 17 \times 19) + 1 = 9699691 = 347 \times 27953$$

but if you go to the *next* prime, its difference from the product is always a prime:

$$5 - 2 = 3$$
$$11 - (2 \times 3) = 5$$
$$37 - (2 \times 3 \times 5) = 7$$
A5235
$$223 - (2 \times 3 \times 5 \times 7) = 13$$
$$2333 - (2 \times 3 \times 5 \times 7 \times 11) = 23$$
$$30047 - (2 \times 3 \times 5 \times 7 \times 11 \times 13) = 17$$
$$510529 - (2 \times 3 \times 5 \times 7 \times 11 \times 13 \times 17) = 19$$
$$9699713 - (2 \times 3 \times 5 \times 7 \times 11 \times 13 \times 17 \times 19) = 23$$

Example 12. From the sequence of primes, form the first differences, then the absolute values of the second, third, fourth,... differences:

2 3 5 7 11 13 17 19 23 29 31 37 41 43 47 53 59 61 67
1 2 2 4 2 4 2 4 6 2 6 4 2 4 6 6 2 6
1 0 2 2 2 2 2 2 4 4 2 2 2 2 0 4 4 2
A36262 1 2 0 0 0 0 0 2 0 2 0 0 0 2 4 0 2
1 2 0 0 0 0 2 2 2 2 0 0 2 2 4 2 2
1 2 0 0 0 2 0 0 0 2 0 2 0 2 2 0
1 2 0 0 2 2 0 0 2 2 2 2 2 0 2 0
1 2 0 2 0 2 0 2 0 0 0 0 2 2 2
1 2 2 2 2 2 2 2 0 0 0 2 0 0 0
1 0 0 0 0 0 0 2 0 0 2 2 0 0
1 0 0 0 0 0 2 2 0 2 0 2 0 0
1 0 0 0 0 2 0 2 2 2 2 2 0

Is the first term in each sequence of differences always 1?

A1567
A50259

Example 13. 2^n is never congruent to 1 (mod n) for $n > 1$. 2^n is congruent to 2 (mod n) whenever n is prime, and occasionally when it isn't ($n = 341, 561, \ldots$). Is 2^n ever congruent to 3 (mod n) for $n > 1$?

Example 14. The good approximations to $5^{1/5}$, namely, the convergents to

$$1 + \frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{2+}\cdots \text{ are } \frac{1}{1}, \frac{3}{2}, \frac{4}{3}, \frac{7}{5}, \frac{11}{8}, \frac{29}{21}, \ldots$$

which have Fibonacci numbers for denominators and Lucas numbers for numerators.

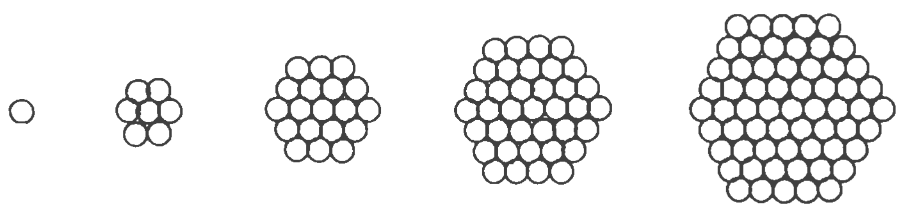
Example 15.

$$(x+y)^3 = x^3 + y^3 + 3xy(x+y)(x^2+xy+y^2)^0$$

$$(x+y)^5 = x^5 + y^5 + 5xy(x+y)(x^2+xy+y^2)^1$$

$$(x+y)^7 = x^7 + y^7 + 7xy(x+y)(x^2+xy+y^2)^2$$

Example 16. The sequence of **hex numbers** (so named to distinguish them from the **hexagonal** numbers, $n(2n-1)$) are depicted in FIG. 2.
The partial sums of this sequence, $1, 8, 27, 64, 125$, appear to be perfect cubes. A578



A3215

1 7 19 37 61

FIG. 2. The hex numbers.

Example 17. Write down the positive integers, delete every second, and form the partial sums of those remaining:

1	~~2~~	3	~~4~~	5	~~6~~	7	~~8~~	9	~~10~~	11
1		4		9		16		25		36

Example 18. As before, but delete every third, then delete every second partial sum:

1	2	~~3~~	4	5	~~6~~	7	8	~~9~~	10	11	~~12~~	13	14	~~15~~	16
1	~~3~~		7	~~12~~		19	~~27~~		37	~~48~~		61	~~75~~		91
1			8			27			64			125			216

Example 19. Again, but delete every fourth, then every third partial sum, then every second of their partial sums:

1	2	3	~~4~~	5	6	7	~~8~~	9	10	11	~~12~~	13	14	15	~~16~~	17
1	3	~~6~~		11	17	~~24~~		33	43	~~54~~		67	81	~~96~~		113
1	~~4~~			15	~~32~~			65	~~108~~			175	~~256~~			369
1				16				81				256				625

Example 20. Again, but circle the first number of the sequence, delete the second after that, the third after that, and so on. Form the partial sums and repeat:

1 2 ~~3~~ 4 5 ~~6~~ 7 8 9 ~~10~~ 11 12 13 14 ~~15~~ 16 17 18 19 20 ~~21~~

2 6 ~~11~~ 18 26 ~~35~~ 46 58 71 ~~85~~ 101 118 136 155 ~~175~~

6 24 ~~50~~ 96 154 ~~225~~ 326 444 580 ~~735~~

24 120 ~~274~~ 600 1044 ~~1624~~

120 720 ~~1764~~

720

Example 21. Write down the odd numbers starting with 43. Circle 43, delete one number, circle 47, delete two numbers, circle 53, delete three numbers, circle 61, and so on. The circled numbers are prime (FIG. 3)

A5846

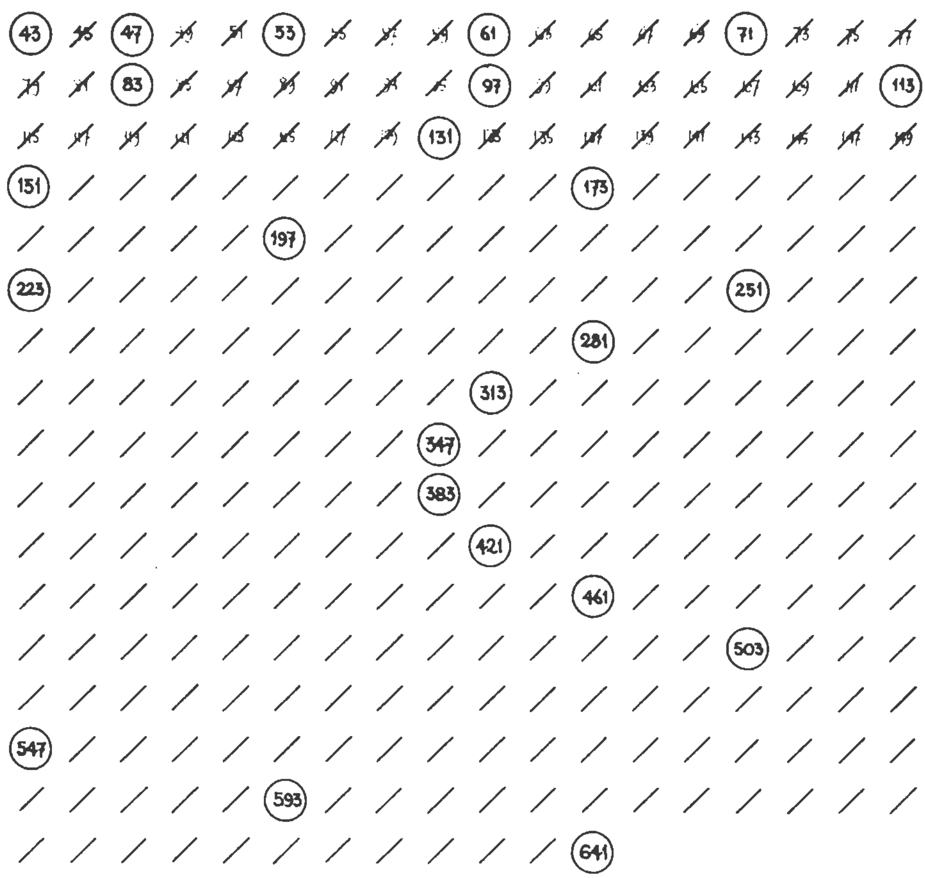


FIG. 3. Parabolas of primes remain.

Example 22. In Table 1 the odd prime values of $n^4 + 1$ and of $17 \times 2^n - 1$ are printed in **bold**. They occur simultaneously for $n = 2, 4, 6, 16, 20$.

TABLE 1

n	$n^4 + 1$	$17 \times 2^n - 1$
0	1	$16 = 2^4$
1	2	$33 = 3 \times 11$
2	**17**	**67**
3	$82 = 2 \times 41$	$135 = 3^3 \times$
4	**257**	**271**
5	$626 = 2 \times$	$543 = 3 \times$
6	**1297**	**1087**
7	$2402 = 2 \times$	$2175 = 3 \times$
8	$4097 = 17 \times$	$4351 = 19 \times$
9	$6562 = 2 \times$	$8703 = 3^2 \times$
10	$10001 = 73 \times$	$17407 = 13^2 \times$
11	$14642 = 2 \times$	$34815 = 3 \times$
12	$20737 = 89 \times$	$69631 = 179 \times$
13	$28562 = 2 \times$	$139263 = 3 \times$
14	$38417 = 41 \times$	$278527 = 223 \times$
15	$50626 = 2 \times$	$557055 = 3^2 \times$
16	**65537**	**1114111**
17	$83522 = 2 \times$	$2228223 = 3 \times$
18	$104977 = 113 \times$	$4456447 = 59 \times$
19	$130322 = 2 \times$	$8912895 = 3 \times$
20	**160001**	**17825791**
21	$194482 = 2 \times$	$35651583 = 3^4 \times$
22	$234257 = 73 \times$	$71303167 = 13 \times$
23	$279842 = 2 \times$	$142606335 = 3 \times$

Example 23. In Table 2 the prime values of $21 \times 2^n - 1$ and of $7 \times 4^n + 1$ are printed in **bold**. They occur simultaneously for $n = 1, 2, 3, 7, 10, 13$.

TABLE 2

n	$21 \times 2^n - 1$	$7 \times 4^n + 1$
0	$20 = 2^2 \times 5$	$8 = 2^3$
1	**41**	**29**
2	**83**	**113**
3	**167**	**449**
4	$335 = 5 \times$	$1793 = 11 \times$
5	$671 = 11 \times$	$7169 = 67 \times$
6	$1343 = 17 \times$	$28673 = 53 \times$
7	**2687**	**114689**
8	$5375 = 5^3 \times$	$458753 = 79 \times$
9	$10751 = 13 \times$	$1835009 = 11 \times$
10	**21503**	**7340033**
11	$43007 = 29 \times$	$29360129 = 37 \times$
12	$86015 = 5 \times$	$117440513 = 3907 \times$
13	**172031**	**469762049**
14	$344063 = 17 \times$	$1879048193 = 11 \times$
15	$688127 = 11^4 \times$	$7516192769 = 29^2 \times$
16	$1376255 = 5 \times$	$30064771073 = 113 \times$
17	$2752511 = 19 \times$	$120259084289 = 379 \times$

Example 24. Consider the sequence

$$x_0 = 1, \qquad x_{n+1} = \left(1 + x_0^2 + x_1^2 + \cdots + x_n^2\right)/(n+1) \qquad (n \geqslant 0).$$

A3504

n	0	1	2	3	4	5	6	7	8	9	...
x_n	1	2	3	5	10	28	154	3520	1551880	267593772160	...

Is x_n always an integer?

new →

Example 25. The same, but with cubes in place of squares: $y_0 = 1$, $y_{n+1} = (1 + y_0^3 + y_1^3 + \cdots + y_n^3)/(n+1)$ $(n \geqslant 0)$. Same question.

A5166

n	0	1	2	3	4	5	...
y_n	1	2	5	45	22815	2375152056927	...

Example 26. Also for fourth powers, $z_{n+1} = (1 + z_0^4 + z_1^4 + \cdots + z_n^4)/(n+1)$.

A5167

n	0	1	2	3	4	...
z_n	1	2	9	2193	5782218987645	...

And for fifth powers, and so on.

Example 27. The irreducible factors of $x^n - 1$ are **cyclotomic polynomials**, i.e., $x^n - 1 = \prod_{d|n}\Phi_d(x)$, so that $\Phi_1(x) = x - 1$, $\Phi_2(x) = x + 1$, $\Phi_3(x) = x^2 + x + 1$, $\Phi_4(x) = x^2 + 1$. The cyclotomic polynomial of order n, $\Phi_n(x)$, has degree $\varphi(n)$, Euler's totient function. It is easy to write down $\Phi_n(x)$ if n is prime, twice a prime, or a power of a prime, and for many other cases. Are the coefficients always ± 1 or 0?

Example 28. If two people play Beans-Don't-Talk, the typical position is a whole number, n, and there are just two options, from n to $(3n \pm 1)/2^*$, where 2^* means the highest power of 2 that divides the numerator. The winner is the player who moves to 1. For example, 7 is a **$\mathscr{P}$-position**, a previous-player-winning position, because the opponent must go to

$$(3 \times 7 + 1)/2 = 11 \text{ or } (3 \times 7 - 1)/2^2 = 5$$

and 11 and 5 are **$\mathscr{N}$-positions**, next-player-winning positions, since they have the options $(3 \times 11 - 1)/2^5 = 1$ and $(3 \times 5 + 1)/2^4 = 1$.

If τ is the probability that a number is an $\mathscr{N}$-position, and there are no $\mathscr{O}$-positions (from which neither player can force a win), then the probability that a number is a $\mathscr{P}$-position is $1 - \tau$. This happens just if both options are $\mathscr{N}$-positions, so $1 - \tau = \tau^2$, and τ is the golden ratio, $(\sqrt{5} - 1)/2 \approx 0 \cdot 618$.

So it is no surprise that 5 out of the first 8 numbers are $\mathscr{N}$-positions, 8 out of the first 13, 13 of the first 21, 21 of the first 34, and 34 of the first 55, since the ratio of consecutive Fibonacci numbers tends to the golden ratio.

Example 29. Does each of the two diophantine equations

$$2x^2(x^2 - 1) = 3(y^2 - 1) \text{ and } x(x-1)/2 = 2^n - 1$$

A180445 have just the five positive solutions $x = 1, 2, 3, 6$, and 91?

Example 30. Consider the sequence $a_1 = 1$, $a_{n+1} = \lfloor \sqrt{2a_n(a_n+1)} \rfloor$ $(n \geq 1)$

A1521

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
a_n	1	2	3	4	6	9	13	19	27	38	54	77	109	154	218	309	437	618	874	1236	1748
			1		2		4		8		16		32		64		128		256		512

Are alternate differences, $a_{2k+1} - a_{2k}$, the powers of two, 2^k?

Example 31. In the same sequence, are the even ranked members, a_{2k+2}, given by $2a_{2k} + \varepsilon_k$, where ε_k is the kth digit in the binary expansion of $\sqrt{2} = 1.01101010000010\ldots$?

Example 32. Is this the same sequence as $a_1 = 1$, $a_2 = 2$, $a_3 = 3$, $a_{n+1} = a_n + a_{n-2}$ $(n \geq 3)$? A930

Example 33. The nth derivative of x^x, evaluated at $x = 1$, is an integer. Is it always a multiple of n? Values for $n = 1, 2, 3, \ldots$ are

$$1 \times 1, 2 \times 1, 3 \times 1, 4 \times 2, 5 \times 2, 6 \times 9, 7 \times (-6), 8 \times 118, 9 \times (-568),$$
$$10 \times 4716, 11 \times (-38160), 12 \times 358126, 13 \times (-3662088), 14 \times 41073096,$$
$$15 \times (-500013528), 16 \times 6573808200, 17 \times (-92840971200),$$
$$18 \times 1402148010528, \ldots.$$

? A5168

Example 34. In how many ways, c_n, can you arrange n pennies in rows, where every penny in a row above the first must touch two adjacent pennies in the row below?

new A5169

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
c_n	1	1	1	2	3	5	9	15	26	45	78	135	234	406	704	1222	2120

To throw more light on such sequences, partition theorists often express their generating function

$$\sum_{n=0}^{\infty} c_n x^n = 1 + x + x^2 + 2x^3 + 3x^4 + 5x^5 + 9x^6 + 15x^7 + \cdots$$

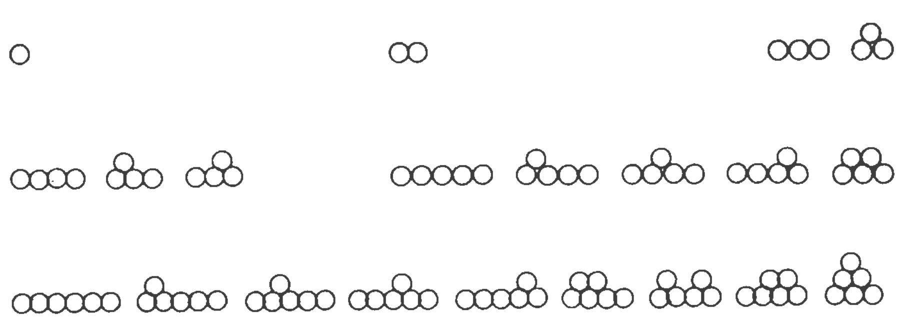


FIG. 4. Propp's penny partitions.

as an infinite product,

$$\prod_{n=1}^{\infty}(1-x^n)^{-a(n)}$$

In this case, $a(n)$ are consecutive Fibonacci numbers:

A5170

n	1	2	3	4	5	6	7	8	9	10	...
$a(n)$	1	0	1	1	2	3	5	8	13	21	...

Example 35. If p_k is the kth prime, $p_1 = 2$, $p_2 = 3, \ldots$, does

A607
A5171

$$\prod_{k=1}^{\infty}(1-x^{p_k})^{-1} = 1 + \sum_{k=1}^{\infty}\frac{x^{p_1+p_2+\cdots+p_k}}{(1-x)(1-x^2)\cdots(1-x^k)}?$$

Answers

1. No less a person than Fermat was fooled by the Strong Law! Euler gave the factorization $2^{32}+1 = 641 \times 6700417$. All other known examples of Fermat numbers are composite; Jeff Young & Duncan Buell [**32**] have recently shown that $2^{2^{20}}+1$ is composite.

A43
A668

2. There are very few **Mersenne primes**, $2^p - 1$. No one can prove that there are infinitely many; $2^{11} - 1 = 23 \times 89$ is not one. See A3 in [**12**] and sequence 1080 in [**28**].

3. In the "prime number race," $4k-1$ and $4k+1$ alternately take the lead infinitely often. This was proved by Littlewood [**18**]. For many papers on this subject see N–12 of *Reviews in Number Theory*, for example, Chen [**4**].

4. A theorem of Legendre (see [**6**], for example) states that if D_+ and D_- are the numbers of divisors of n of shapes $4k+1$ and $4k-1$, then the number of representations of n as the sum of two squares is $4(D_+ - D_-)$. So $D_+ \geqslant D_-$ for every number!

5. Before we reveal all, here is a circle (FIG. 5) with ten points to further confuse you. It has 256 regions.

If the circle has n points, there are $\binom{n}{4}$ intersections of chords inside the circle, since each set of four points gives just one such intersection. The number of vertices in the figure is $V = n + \binom{n}{4}$. To find the number of edges, count their ends. There are $n+1$ at each of the n points and four at each of the $\binom{n}{4}$ intersections, so $2E = n(n+1) + 4\binom{n}{4}$. By Euler's formula, the number of regions inside the circle is

$$\begin{aligned}
E + 1 - V &= 2\binom{n}{4} + \frac{1}{2}n(n+1) + 1 - \binom{n}{4} - n \\
&= \binom{n}{4} + \frac{1}{2}n(n-1) + 1 \\
&= \binom{n-1}{4} + \binom{n-1}{3} + \binom{n-1}{2} + \binom{n-1}{1} + \binom{n-1}{0}.
\end{aligned}$$

A direct proof, by labelling the regions with at most four of the numbers $1, 2, \ldots,$

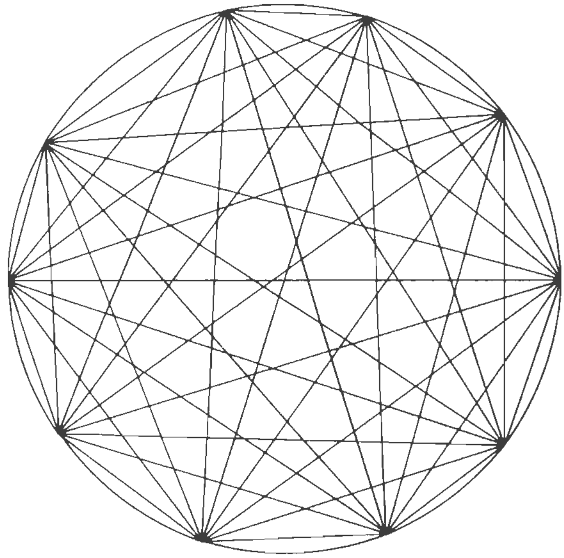


FIG. 5. Circle partitioned into 2^8 regions.

$n - 1$, will appear in [5]. The answer is just five of the n terms in the binomial expansion of $(1 + 1)^{n-1}$. For $n < 6$, this is all the terms, and the number is a power of 2. For $n = 6$, only 1 is missing. For $n = 10$ just half the terms are missing, and the number of regions is $\frac{1}{2} \cdot 2^9 = 256$.

# of points =	1	2	3	4	5	6	7	8	9	10	11	12	13	14
# of regions =	**1**	**2**	**4**	**8**	**16**	31	57	99	163	**256**	386	562	794	1093

A127

Some other famous numbers, e.g. 163 and 1093, also occur in this sequence, number 427 in [28]. A127

6. No member of this sequence is divisible by 2, 3, 5, 7, 11, 13, or 37, as may be seen immediately from well known divisibility tests. On the other hand, 17, 19, 23, 29, 31,... divide 33...331 just if the number of threes is respectively $16k + 8$, $18k + 11$, $22k + 20$, $28k + 19$, $15k + 1, \ldots$, while 41, 43, 53, 67, 71, 73, 79,... divide no members of the sequence. I don't think that there is a simple description of which primes do, and which primes don't, divide. The next member, 33333331, is also prime, but $333333331 = 17 \times 19607843$

7. We've again given ourselves a good start, since $\sum_{k=1}^{n}(-1)^{n-k}k!$ is not divisible by any prime $\leqslant n$. However,

$$9! - 8! + 7! - 6! + 5! - 4! + 3! - 2! + 1! = 326981 = 79 \times 4139.$$

8. This example, as well as example **5**., was first shown to me by Leo Moser, a quarter of a century ago. Row n is the list of denominators of the **Farey series** of order n, i.e., the set of rational fractions r, $0 \leqslant r \leqslant 1$, whose denominators do not exceed n. In getting row n from row $n - 1$, just $\varphi(n)$ numbers are inserted, where $\varphi(n)$ is Euler's totient function, the number of numbers not exceeding n which are prime to n. It is fortuitous that $1 + \sum_{k=1}^{n}\varphi(k)$ is prime for $1 \leqslant n \leqslant 9$. As $\varphi(10) = 4$, the number of numbers in row 10 is $29 + 4 = 33$, and is not prime.

9. The expression $7013 \times 2^n + 1$ is composite for $0 \leqslant n \leqslant 24160$ [**15**]. Duncan Buell & Jeff Young have sieved out 325 further candidates $n < 10^5$ which might yield a prime. None is known, though it's likely that there is one.

10. The number $78557 \times 2^n + 1$ is always divisible by at least one of $3, 5, 7, 13, 19, 37, 73$ [**26, 27**]. For this and the previous example, see also B21 in [**12**].

11. R. F. Fortune conjectured that these differences are always prime: see [**8**], [**9**]

A5235 and A2 in [**12**]. The next few are 37, 61, 67, 61, 71, 47, 107, 59, 61, 109, 89, 103, 79. There's a high probability that the conjecture is true, because the difference can't be divisible by any of the first k primes, so the smallest composite candidate for $P = \prod p_k$ is p_{k+1}^2, which is approximately $(k \ln k)^2$ in size. The product of the first k primes is about e^k: to find a counter example we need a gap in the primes near N of size at least $(\ln N \ln\ln N)^2$. Such gaps are believed not to exist, but it's beyond our present means to prove this.

12. This is N. L. Gilbreath's conjecture, which has been verified for $k < 63419$ [**16**]. Hallard Croft has suggested that it has nothing to do with primes as such, but will be true for any sequence consisting of 2 and odd numbers, which doesn't increase too fast, or have too large gaps: A10 in [**12**]. In an 87-08-03 letter, Andy Odlyzko reported that he had verified the conjecture for $k < 10^{10}$.

A50259 **13**. D. H. & Emma Lehmer discovered that $2^n \equiv 3 \pmod{n}$ for $n = 4700063497$, but for no smaller $n > 1$.

14. The kth Lucas number and the $(k+1)$th Fibonacci number are

$$\left(\frac{1+\sqrt{5}}{2}\right)^k + \left(\frac{1-\sqrt{5}}{2}\right)^k \quad \text{and} \quad \frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^{k+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{k+1}\right\}.$$

Their ratio, as k gets large, approaches $(5-\sqrt{5})/2 \approx 1 \cdot 381966011$, whereas $5^{1/5} \approx 1 \cdot 379729661$. The next few convergents to $5^{1/5}$,

A45
A2363

$$\frac{40}{29}, \frac{109}{79}, \frac{912}{661}, \frac{1021}{740}, \frac{26437}{19161}, \frac{27458}{19901},$$

A204 do not involve Fibonacci or Lucas numbers. Compare sequences 256 & 260 and 924 & 925 in [**28**]. This example goes back to 1866 [**25**].

A2364 **15**. This is quite fortuitous [**30**]. Put $x = y = 1$, giving $2^{2n+1} - 2 = (2n+1) \times 2 \times 3^{n-1}$. It's true that

$$2^2 - 1 = 3 \times 3^0, \qquad 2^4 - 1 = 5 \times 3^1, \qquad 2^6 - 1 = 7 \times 3^2$$

but it's clear that the pattern can't continue.

16. The $(n+1)$th hex number, $1 + 6 + 12 + \cdots + 6n = 3n^2 + 3n + 1$, when added to n^3, gives $(n+1)^3$, so the pattern is genuine. It is instructive to regard the nth hex number as comprising the three faces at one corner of a cubic stack of n^3 unit cubes (FIG. 6).

17, **18**, **19**, and **20** are examples of **Moessner's process**, which does indeed produce the square, cubes, fourth powers and factorials. Moessner's paper [**20**] is followed by a proof by Perron. Subsequent generalizations are due to Paasche [**22**]: see [**19**] for a more recent exposition.

21. A thinly disguised arrangement of Euler's formula, $n^2 + n + 41$, which gives

A5846 primes for $-40 \leqslant n \leqslant 39$. For $n = 40$, $n^2 + n + 41 = 41^2$. See A1 and Fig. 1 in

etc [**12**]. For remarkable connexions with quadratic fields, continued fractions, modular functions and class numbers, see [**29**].

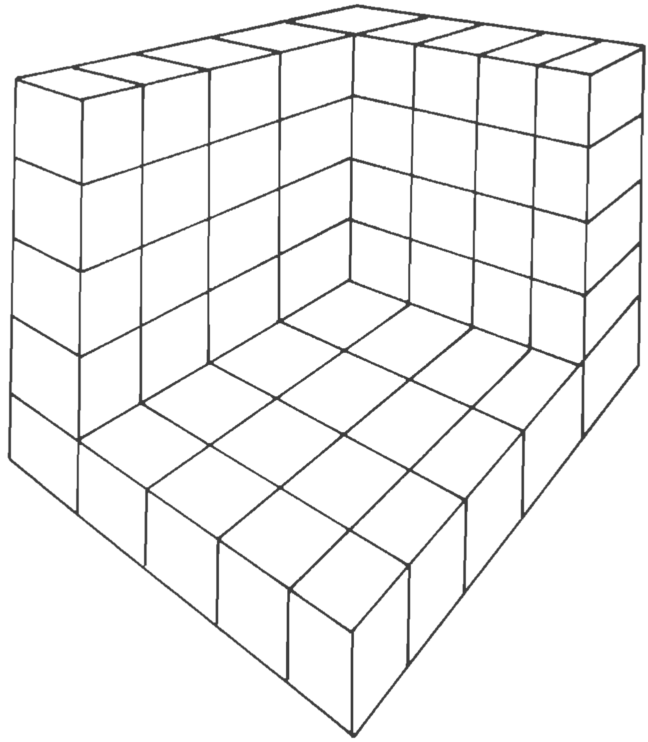


FIG. 6. The fifth hex number.

22. The initial pattern is explained by the facts that if n is odd, $n^4 + 1$ is even, and $17 \times 2^n - 1$ is a multiple of 3. Thereafter it's largely coincidence until $n = 24$, for which $n^4 + 1 = 331777$ is prime, while $17 \times 2^n - 1 = 285212671 = 149 \times 1914179$. See [**17**], [**24**] and sequences 386 and 387 in [**28**]. A68 A1774

23. This is also a coincidence, until we reach $n = 18$, for which $21 \times 2^n - 1 = 5505023$ is prime, while

$$7 \times 4^n + 1 = 481036337153 = 166609 \times 2887217.$$ A2238

See [**31**], [**23**] and sequences 314 & 315 in [**28**]. A2255

24. A sequence introduced by Fritz Göbel. A more convenient recursion for calculation is $(n+1)x_{n+1} = x_n(x_n + n), (n \geqslant 1)$. If you work modulo 43, you'll find that for A3504

$n =$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
$x_n \equiv$	1	2	3	5	10	28	25	37	10	20	15	38	19	42	36	34	2	35	39	31	13	2

A259878

$n =$	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42
$x_n \equiv$	6	26	28	29	4	14	42	5	20	17	4	20	16	29	42	13	42	20	8	23	33

and $x_{42}(x_{42} + 42) \equiv -10(-10 + 42) = -320$, which is not divisible by 43, so x_{43} is not an integer, although x_n is an integer for $0 \leqslant n \leqslant 42$.

25. Similar calculations, mod 89, using the relation $(n+1)y_{n+1} = y_n(y_n^2 + n)$, show that y_{89} is not an integer. For this, and the previous example, see E15 is **[12]**.
26. Since this question was asked, Henry Ibstedt has made extensive calculations, and found the first noninteger term, x_n, in the sequence involving kth powers, to be

A108394

k	2	3	4	5	6	7	8	9	10	11
n	43	89	97	214	19	239	37	79	83	239

He also found corresponding results with different initial values. The longest to hold out ($n = 610$) are the cubes ($k = 3$, Example **25**) with $x_0 = 1$, $x_1 = 11$.
27. The first cyclotomic polynomial to display a coefficient other than ± 1 and 0 is

A13594

$$\begin{aligned}\Phi_{105}(x) = x^{48} + x^{47} + x^{46} - x^{43} - x^{42} - 2x^{41} - x^{40} - x^{39} + x^{36} + x^{35} + x^{34}\\ + x^{33} + x^{32} + x^{31} - x^{28} - x^{24} - x^{22} - x^{20} + x^{17} + x^{16} + x^{15} + x^{14}\\ + x^{13} + x^{12} - x^{9} - x^{8} - 2x^{7} - x^{6} - x^{5} + x^{2} + x + 1\end{aligned}$$

Coefficients can be unboundedly large, but require n to contain a large number of distinct odd prime factors; see **[8]**. More recently, Montgomery & Vaughan **[33]** have shown that if $\Phi_n = \Sigma a(m,n)x^m$ and $L(n) = \ln \max_n |a(m,n)|$ then, for m large, $\frac{m^{1/2}}{(\ln 2m)^{1/4}} \ll L(n) \ll \frac{m^{1/2}}{(\ln m)^{1/4}}$.
28. This game was misremembered by John Conway from John Isbell's game of Beanstalk **[13]**. The Fibonacci pattern is not maintained: only 52 of the first 89 numbers, 81 of the first 144, 126 of the first 233, and 201 of the first 377, are $\mathcal{N}$-positions. The probability argument is fallacious: the probabilities of the status of the two options are *not* independent.
29. True, but why the coincidence?
30 and **31**. The patterns of powers of 2 and of binary digits of $\sqrt{2}$ both continue; see **[11]**, **[14]** and sequence 206 in **[28]**.
A1521
A930
32. A different sequence, number 207 in **[28]**, which agrees for $n < 9$, but then continues 28, 41, 60, 88, 129, 189, 277, 406, 595, 872, 1278, ...
33. If $y = x^x$ and $y_n(1)$ denotes the value of d^ny/dx^n at $x = 1$, then

$$y_{n+1}(1) = y_n(1) + \binom{n}{1}y_{n-1}(1) - \binom{n}{2}y_{n-2}(1) + 2!\binom{n}{3}y_{n-3}(1) - 3!\binom{n}{4}y_{y-4}(1)$$
$$+ - + \cdots + (-1)^n(n-1)!.$$

This was not known to be a multiple of $n+1$ when it was submitted to the Unsolved Problems section of this MONTHLY by Richard Patterson & Gaurar Suri. But in an 87-05-28 letter, Herb Wilf gives a proof, using the generating function for Stirling numbers of the first kind. His proof in fact shows that $n(n-1)$ divides $y_n(1)$ just if $n-1$ divides $(n-2)!$, which it does for $n \geqslant 7$, provided that $n-1$ is not prime.
34. This sequence was investigated by Jim Propp. Except that $a(12) = 55$, the pattern of Fibonacci numbers does not continue:

$n =$ 11	12	13	14	15	16	17	18
$a(n) =$ 35	55	93	149	248	403	670	1082

Since this was written, Wilf **[21]** has linked the generating function with Ramanujan's continued fraction, and he observes that the numbers of propper partitions with k coins in the lowest row are yet another manifestation of the Catalan numbers, 1, 2, 5, 14, 42, ... [7]. These partitions are a variant of some considered by Auluck [1]. Auluck's partitions have the pennies contiguous in *every* row, not just the lowest. Their numbers 1, 1, 2, 3, 5, 8, ... are another good example of the Strong Law.

35. The expansion of the product as a power series, is

$$1 + x^2 + x^3 + x^4 + 2x^5 + 2x^6 + 3x^7 + 3x^8 + 4x^9 + 5x^{10} + 6x^{11} + 7x^{12} + 9x^{13}$$
$$10x^{14} + 12x^{15} + 14x^{16} + 17x^{17} + 19x^{18} + 23x^{19} + 26x^{20} + 30x^{21} + 35x^{22}$$
$$+40x^{23} + 46x^{24} + 52x^{25} + 60x^{26} + 67x^{27} + 77x^{28} + 87x^{29} + \cdots .$$

A607

The sum is the same, until... $+31x^{21} + 35x^{22}$

$$+41x^{23} + 46x^{24} + 54x^{25} + 60x^{26} + 69x^{27} + 78x^{28} + 89x^{29} + \cdots .$$

A5171

This was entry 29 in Chapter 5 of Ramanujan's second notebook [2], [3]: but he had crossed it out!

Let me know if I've missed out your favorite example!

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