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# Modular forms on non-compact arithmetic triangle groups

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#### Introduction

The connections between binary, ternary and quaternary error correcting codes on the one hand, and lattices in  $\mathbb{R}^n$  on the other have been studied by many authors. In their study, the knowledge of the theory of modular forms of lower levels plays an important role. In return, by their results, we can understand the ring of modular forms on  $SL_2(\mathbb{Z})$  completely through the plynomial ring generated by weight enumerators of self-dual doubly even binary linear codes. This map is called Brouè- Enguehard map. This clarifies the meaning of the expression of modular forms in terms of Jacobi theta functions.

The connection between the arithmetic-geometric means and hypergeometric functions was already known by Gauss. Only ten years ago, J.Borwein and P.Borwein found a cubic analogue of the arithmetic-geometric mean. In their study, the fact that certain modular forms on  $\Gamma_0(2)$ ,  $\Gamma_0(3)$  and  $\Gamma_0(4)$  are expressed in terms of hypergeometric functions, plays an important role. This fact was already known by Ramanujan.

The common feature of these two connections are modular forms on non-compact arithmetic triangle groups. Modular forms on some of these groups have appeared in other areas like as Hecke groups and Mirror maps.

Takeuchi showed that there are nine such groups. In this paper, we study holomorphic modular forms on all these groups. We shall show that we can deal with modular forms on these groups in a unified manner using hypergeometric functions. We shall give a basis of the space of holomorphic modular forms of every even weights explicitly in terms of hypergeometric functions and modular functions. As a result, we obtain generalizations of Brouè-Enguehard map. Our results have many applications. We can study differential equations which have modular forms on these groups as their solutions. We will discuss this result in a subsequent paper.

The outline of this paper is as follows. In Section 1, we summarize the results from codes, weight enumerators and lattices that will be needed in the sequel. In Section

summarize properties of hypergeometric functions associated to non-compact arithmetic triangle groups.

In the remaining sections, we devote to construct a basis of the space of holomorphic modular forms of every even weights on these groups.

## 1 Code, Weight enumerator and Modular forms

We recall several facts from Coding theory.

A code C of length n over the field  $\mathbf{F}_q$ ,  $q = p^a$  (p prime), consists of a set of n-tuples  $\mathbf{u} = (u_1, u_2, \dots, u_n), u_i \in \mathbf{F}_q$ . C is called linear if C form a vector space over  $\mathbf{F}_q$ . The Hamming weight of a vector  $\mathbf{u} = (u_1, u_2, \dots, u_n)$ , written  $wt(\mathbf{u})$ , is the number of non zero  $u_i$ . The Hamming weight enumerator of C is

$$W_C(X,Y) = \sum_{\mathbf{u} \in C} X^{n-wt(\mathbf{u})} Y^{\mathbf{u}}.$$

For  $\mathbf{u}, \mathbf{v} \in \mathbf{F}_q^n$ , we define their scalar product by  $\mathbf{u} \cdot \mathbf{v} = \sum u_i v_i$ . The dual C of a linear code C consists of all vectors  $\mathbf{v} \in \mathbf{F}_q^n$  such that  $\mathbf{u} \cdot \mathbf{v} = 0$  for all  $\mathbf{u} \in C$ . A linear code C is called doubly even if the weight of all codewords in C are divisible by 4.

A code over  $F_2, F_3, F_4$  is called binary, ternary, quaternary respectively.

Sloane [16], Conway-Sloane [5]

We consider binary codes.

Theorem 1.1 (Gleason) Let C be a binary linear code which is self-dual and doubly even.

Then

$$W_C(x,y) \in \mathbb{C}[f_1(x,y), f_2(x,y)],$$

where 
$$f_1(x,y) = x^8 + 14x^4y^4 + y^8, f_2(x,y) = x^4y^4(x^4 - y^4)^4$$
.

From binary linear codes, we can construct lattices. We take the standard lattice  $\mathbb{Z}^n \subset \mathbb{R}^n$  and consider the reduction mod 2:

$$\rho: \mathbf{Z}^n \to \mathbf{F}_2^n$$
.

Let C be a linear code in  $\mathbf{F}_2^n$ . Then  $\rho^{-1}(C)$ , the preimage of C in  $\mathbf{Z}^n$ , is a lattice in  $\mathbf{R}^n$ . We define  $\Lambda_C = \frac{1}{\sqrt{2}}\rho^{-1}(\mathsf{C})$ . The theta function of the lattice  $\Lambda_C$  is defined by

$$\Theta_{\Lambda_C}(\tau) = \sum_{x \in \Lambda_C} e^{\pi i \tau \langle x, x \rangle}.$$

We express the theta functions of lattices in terms of the following Jacobi theta functions:

$$\theta_3(\tau) = 1 + 2 \sum_{n=1}^{\infty} q^{(1/2)n^2},$$

$$\theta_2(\tau) = 2 \sum_{n=0}^{\infty} q^{(1/2)(n^2+n+1/4)}$$

Then the following proposition is easily proved and is well known.

PROPOSITION 1.2 Let C be a binary linear code with Hamming weight enumerator  $W_C(x, y)$ . Then

$$\Theta_{\Lambda_C}(\tau) = W_{\mathsf{C}}(\theta_3(2\tau), \theta_2(2\tau)).$$

We define the map  $\Phi_2: \mathbf{C}[x,y] \to \mathbf{C}[[q]]$  by



$$\Phi_2(f(x,y)) = f(\theta_3(\tau), \theta_2(\tau)).$$

THEOREM 1.3 (Broue-Enguehard)  $\Phi_2$  gives the isomorphism from  $C[f_1(x,y), f_2(x,y)]$  to  $C[E_4, \Delta]$ , and  $\Phi_2(f_1(x,y)) = E_4(\tau), \Phi_2(f_2(x,y)) = 16\Delta(\tau)$ .

We consider ternary codes.

Theorem 1.4 (Gleason) Let C be a ternary linear code which is self-dual. Then

$$W_C(x,y) \in C[g_1(x,y), g_2(x,y)],$$

where 
$$g_1(x,y) = x^4 + 8xy^3, g_2(x,y) = y^3(x^3 - y^3)^3$$
.

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From ternary linear codes, we can construct complex lattices.Let O be the ring of integers in the imaginary quadratic field  $Q(\sqrt{-3})$ . Let  $(\pi)$  denote the prime ideal of norm 3. Then  $(O/\pi O)F_3$ . We take the standard lattice  $O^n$  in  $C^n$  and consider reduction mod  $(\pi)$ :

$$\rho: \mathbf{O}^n \to \mathbf{F}_3^n$$
.

Let C be a ternary linear code. We define  $\Lambda_C = \rho^{-1}(C)$ , which is a lattice in  $\mathbf{O}^n$ . The theta function of the lattice  $\Lambda_C$  is defined by

$$\Theta_{\Lambda_C}(\tau) = \sum_{x \in \Lambda} e^{2\pi i \tau \langle x, x \rangle/3}.$$

We can express the theta function in terms of Jacobi theta functions too.

PROPOSITION 1.5 Let C be a ternary linear code with Hamming weight enumerator  $W_C(X,Y)$ . Then

$$\Theta_{\Lambda}(\tau) = W_{\mathsf{C}}(\psi_0(\tau), \psi_1(\tau)).$$

where

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We define the map  $\Phi_3: \mathbf{C}[x,y] \to \mathbf{C}[[q]]$  by

$$\Phi_3(f(x,y)) = f(\psi_0(\tau), \psi_1(\tau)).$$

THEOREM 1.6 (Broue-Enguehard)  $\Phi_3$  gives the isomorphism from  $C[g_1(x,y), g_2(x,y)]$  to  $C[E_4, \Delta]$ , and  $\Phi_3(g_1(x,y)) = E_4(\tau), \Phi_3(g_2(x,y)) = 27\Delta(\tau)$ .

We consider quaternary codes.

THEOREM 1.7 (MacWilliams-Mallows-Sloane) Let C be a quaternary linear code which is self-dual. Then

$$W_C(x,y) \in C[h_1(x,y), h_2(x,y)],$$

where 
$$h_1(x,y) = x^2 + 3y^2, h_2(x,y) = y^2(x^2 - y^2)^2$$
.

From quaternary linear codes, we can construct complex lattices too. Let (2) denote the prime ideal of norm 4. Then  $(O/2O)F_4$ . We take the standard lattice  $O^n$  in  $C^n$  and consider reduction mod (2):

$$\rho_4: \mathrm{O}^n \to \mathrm{F}_4^n$$
.

Let C be a quaternary linear code. We define  $\Lambda_C = \rho_4^{-1}(C)$ , which is a lattice in  $O^n$ . The theta function of the lattice  $\Lambda_C$  is defined by

$$\Theta_{\Lambda_C}(\tau) = \sum_{x \in \Lambda_C} e^{2\pi i \tau \langle x, x \rangle}.$$

We can express the theta function in terms of Jacobi theta functions too(For the proof, see [17]).

PROPOSITION 1.8 Let C be a quaternary linear code with Hamming weight enumerator  $W_C(X,Y)$ . Then

$$\Theta_{\Lambda_C}(\tau) = W_{\mathsf{C}}(\phi_0(\tau), \phi_1(\tau)).$$

where

$$\phi_0(\tau) = \psi_0(\tau), \qquad \qquad \phi_0(\tau) = \theta_2(2\tau)\theta_3(6\tau) + \theta_3(2\tau)\theta_2(6\tau).$$

$$\phi_1(\tau) = \theta_2(2\tau)\theta_3(6\tau) + \theta_3(2\tau)\theta_2(6\tau).$$



$$\Phi_4(f(X,Y)) = f(\phi_0(\tau), \phi_1(\tau)).$$

It is natural to consider  $\Phi_4(x^2+3y^2)$ ,  $\Phi_4(y^2(x^2-y^2)^2)$ . Contrary to the cases of binary and ternary codes, they are not modular forms on  $SL(2, \mathbf{Z})$ . We shall express these modular forms in terms of modular forms on certain non-compact, arithmetic triangle groups.

## 2 Hypergeometric functions associated with non-compact arithmetic triangle groups

Let  $\alpha, \beta, \gamma$  be complex numbers which are not zero. Hypergeometric function  $F(\alpha, \beta; \gamma; z)$  is defined by

$$F(\alpha, \beta; \gamma; z) = \sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n}{(1)_n(\gamma)_n} x^n$$

where  $(\alpha)_0 = 1, (\alpha)_n = \alpha(\alpha+1)\cdots(\alpha+n-1)(n \ge 1)$ .

The Eisenstein series  $E_4(\tau)$ ,  $E_6(\tau)$  are expressed by the hypergeometric function as follows (For the proof, see Stiller [18]):

(1) 
$$E_4(\tau) = F(\frac{1}{12}, \frac{5}{12}, 1, \frac{12^3}{j(\tau)})^4,$$

(2) 
$$E_6(\tau) = (1 - \frac{12^3}{j(\tau)})^{\frac{1}{2}} F(\frac{1}{12}, \frac{5}{12}, 1, \frac{12^3}{j(\tau)})^6.$$

Hence we get

$$\Delta(\tau) = \frac{E_4(\tau)^3 - E_6(\tau)^2}{12^3},$$

$$= \frac{1}{j(\tau)} F(\frac{1}{12}, \frac{5}{12}, 1, \frac{12^3}{j(\tau)})^{12}.$$

Put  $F = F(\frac{1}{12}, \frac{5}{12}, 1, \frac{12^3}{j(\tau)})$  and  $d = \dim_{\mathbb{C}} M_k$ . Then, if  $k \equiv 0 \pmod{4}$ ,

$$\left\{F^k, \frac{1}{j(\tau)}F^k, \cdots, \frac{1}{j(\tau)^{d-1}}F^k\right\}$$

is a basis of  $M_k$ , and if  $k \equiv 2 \pmod{4}$ ,

$$\left\{ (1 - \frac{12^3}{j(\tau)})^{1/2} F^k, (1 - \frac{12^3}{j(\tau)})^{1/2} \frac{1}{j(\tau)} F^k, \cdots, (1 - \frac{12^3}{j(\tau)})^{1/2} \frac{1}{j(\tau)^{d-1}} F^k \right\}$$

is a basis of  $M_k$ . A modular form on  $SL(2, \mathbb{Z})$  is written as a linear sum of these explicit modular forms expressed in terms of the hypergeometric function and the modular function.

Takeuchi [19] showed that there exist nine non-compact arithmetic triangle groups. The famous one among them is the modular group  $SL(2, \mathbf{Z})$ . We shall show that modular forms on these nine groups are also written by linear sums of explicit modular forms which are expressed in terms of hypergeometric functions and modular functions.

The following table gives the nine non-compact arithmetic triangle groups:

Table 1

1A	2,3,∞	$\left(\frac{1}{12},\frac{5}{12},1\right)$	$\Gamma_0(1)$
2A	2,4,∞	$\left(\frac{1}{8},\frac{3}{8},1\right)$	Γ*(2)
2В	$2,\infty,\infty$	$\left(\frac{1}{4},\frac{3}{4},1\right)$	$\Gamma_0(2)$
3A	2,6,∞	$\left(\frac{1}{6},\frac{1}{3},1\right)$	Γ*(3)
3В	$3,\infty,\infty$	$\left(\frac{1}{3},\frac{2}{3},1\right)$	$\Gamma_0(3)$
4C	$\infty,\infty,\infty$	$(\frac{1}{2},\frac{1}{2},1)$	$\Gamma_0(4)$
2a.	3,3,∞	$\left(\frac{1}{6},\frac{1}{2},1\right)$	
4a	4,4,∞	$\left(\frac{1}{4},\frac{1}{2},1\right)$	
6a	6,6,∞	$\left(\frac{1}{3},\frac{1}{2},1\right)$	

Conway-Norton [4] showed that a lots of modular function of genus zero are parametrized by conjugacy classes of the Monster group. Afterthat Ford-Mckay-Norton[8] extended the list of modular functions of genus zero. First column gives their name.

The second column gives ramification indices. The third column gives the corresponding hypergeometric functions  $F(\alpha, \beta; \gamma; z)$ .

The fourth column gives the usual notation of arithmetic congruence subgroups. The blank means that they are not congruence subgroups.

The modular function field with respect to these groups are genus zero and we can see the normalized generator  $T_g(\tau)$ , which is called Thompson series, in Conway-Norton [4] and Ford-Mckay-Norton[8]. But the constant term of Thompson series are determined to be zero. For our purpose, we need to choose exact constant, which are given in the following table with other Fourier coefficients. We denote these modular functions by  $t_g(\tau)$ .

Table 2

	$t_{1A}$	$t_{2A}$	$t_{2B}$	$t_{4C}$	$t_{3A}$	$t_{3B}$			
-1	1	1	1	1	1	1			
0	744	104	40	8	42	15			
1	196884	4372	276	20	783	54			
2	21493760	96256	-2048	0	8672	-76			
3	864299970	1240002	11202	-62	65367	-243			
4	20245856256	10698752	-49152	0	371520	1188			
5	333202640600	74428120	184024	216	1741655	-1384			
6	4252023300096	431529984	-614400	0	7161696	-2916			
7	44656994071935	2206741887	1881471	-641	26567946	11934			
8	401490886656000	10117578752	-5373952	0	90521472	-11580			
9	3176440229784420	42616961892	14478180	1636	288078201	-21870			
10	22567393309593600	166564106240	-37122048	0	864924480	79704			
11	146211911499519294	611800208702	91231550	-3778	2469235686	-71022			
	A 521	A7267	A35099	,	1	P			
Ea	ch row with n gives Fo	urier coefficient	of $q^n$ .		A30197				
		) 1130111			A 198 9 55				
A29845									
							•		

M-Tains of class 4a

1 A7250 - A7260  $t_{2a}$  $6\sqrt{-3}$  $24\sqrt{-3}$  $16\sqrt{-1}$ 0 -76 -33  $\frac{1}{2}$ -492  $\frac{3}{2}$ -702 -153 -22590 -713  $\frac{5}{2}$ -5224 -367400  $\frac{7}{2}$ -23425 -2550-3764865  $\frac{9}{2}$ -98172 -7479 -28951452  $\frac{11}{2}$ -20314 -182474434 -336450  $\frac{13}{2}$ -51951 -990473160 -1094152  $\frac{15}{2}$ -122229 -3188349 -4780921725  $\frac{17}{2}$ -276656 -20974230680 -8913752  $\frac{19}{2}$ -84963769662 -23247294 -601068

Each row with  $\frac{n}{2}$  gives Fourier coefficient of  $q^{\frac{n}{2}}$ .

These modular functions have already appeared in many literatures. We quote a few of them, J.Borwein and P. Borwein [2], and B. Lian and S.-T. Yau [13]. Triangular replicable functions considered in M. Hanrad and J. Mckay [9] are the same as above. They satisfy algebraic relations. In the following table, we write them in a unified form: The relation is

$$\frac{c}{t} = f(\frac{c'}{t'}).$$

t	$t$ c $t_{1A}$ 1728 $t_{1A}$ 1728			f(x)		t'		c'	
$t_1$			3	$\frac{64x(1-x)^3}{(1+8x)^3}$		$t_{3B}$		27	
$t_{1A}$				$\frac{27x(1-x)^2}{(1+3x)^3}$		$t_{2B}$		64	
$t_{2B}$		64		$\frac{4x}{(1+x)^2}$		$t_{4C}$		16	
$t_{2A}$	t <sub>2A</sub> 256			-4x(x-1)		$t_{2B}$		64	
$t_{3A}$		108		-4x(x-1)	1	3 <i>B</i>		27	
$t_{1A}$	1728			$\frac{x^2}{4(x-1)}$	t	2a	4	$8\sqrt{-3}$	
$t_{3A}$	A 108			$\frac{x^2}{4(x-1)}$	t	6a	1:	$2\sqrt{-3}$	
$t_{2A}$	2	256		$\frac{x^2}{4(x-1)}$	t.	1a	32	$2\sqrt{-1}$	

If we can find the forms of these algebraic relations, it is easy to prove them. So we omit the proof. Some of them are already appeared in the above quoted literature.

We notice that these algebraic relations between  $t_g(\tau)$  are closely related to the quadratic, cubic and quintic transformations of the hypergeometric functions. We use the following

quadratic transformations of the hypergeometric functions which are well-known in the book of Erdélyi [7]:

(3) 
$$F(a,b;2b;z) = (1-z)^{-a/2}F(\frac{1}{2}a,b-\frac{1}{2}a;b+\frac{1}{2};\frac{-z^2}{4(1-z)}),$$

(4) 
$$F(a,b;\frac{a+b+1}{2};z) = F(\frac{1}{2}a,\frac{1}{2}b;\frac{a+b+1}{2};4z(1-z)).$$

PROPOSITION 2.1

(5) 
$$F(\frac{1}{12}, \frac{5}{12}; 1; \frac{12^3}{j(\tau)}) = (1 - \frac{48\sqrt{-3}}{t_{2a}})^{1/12} F(\frac{1}{6}, \frac{1}{2}; 1; \frac{48\sqrt{-3}}{t_{2a}}),$$

(6) 
$$F(\frac{1}{8}, \frac{3}{8}; 1; \frac{258}{t_{2A}}) = (1 - \frac{32\sqrt{-1}}{t_{4a}})^{1/8} F(\frac{1}{4}, \frac{1}{2}; 1; \frac{32\sqrt{-1}}{t_{4a}}),$$

(7) 
$$F(\frac{1}{6}, \frac{1}{3}; 1; \frac{108}{t_{3A}}) = (1 - \frac{12\sqrt{-3}}{t_{6a}})^{1/6} F(\frac{1}{3}, \frac{1}{2}; 1; \frac{12\sqrt{-3}}{t_{6a}}).$$

PROOF. This follows from (3) and Table 3.

#### PROPOSITION 2.2

$$F(\frac{1}{4}, \frac{3}{4}; 1; \frac{64}{t_{2B}}) = F(\frac{1}{8}, \frac{3}{8}; 1; \frac{256}{t_{2A}}),$$
  
$$F(\frac{1}{3}, \frac{2}{3}; 1; \frac{27}{t_{3B}}) = F(\frac{1}{6}, \frac{1}{3}; 1; \frac{108}{t_{3A}}).$$

PROOF. This follows from (4) and Table 3.

It is known that some of our  $t_g$  are expressed in terms of the Jacobi theta functions. We refer [2] [6] [15].

(8) 
$$t_{2B} = 16 \frac{(\theta_3(2\tau)^4 + \theta_2(2\tau)^4)^2}{\theta_3(2\tau)^4 \theta_2(2\tau)^4},$$

(9) 
$$t_{4C} = 16 \frac{\theta_3 (2\tau)^4}{\theta_2 (2\tau)^4},$$

(10) 
$$t_{3B} = 27 \frac{\psi_0(\tau)^3}{\psi_1(\tau)^3}.$$

Hence we can restate Theorems 2.2,2.3 and 2.6 in [2] in terms of Thompson series.

#### THEOREM 2.3

(11) 
$$F(\frac{1}{4}, \frac{3}{4}; 1; \frac{64}{t_{2B}})^2 = \theta_3(2\tau)^4 + \theta_2(2\tau)^4,$$

(12) 
$$F(\frac{1}{2}, \frac{1}{2}; 1; \frac{16}{t_{4C}})^2 = \theta_3(2\tau)^4,$$

(13) 
$$F(\frac{1}{3}, \frac{2}{3}; 1; \frac{27}{t_{3B}})^2 = \psi_0(\tau)^2.$$

#### 2.1 Poincaré series of the ring of modular forms

Let  $\Gamma$  be one of nine non-compact, arithmetic triangle groups. Let k be an even, non-negative integer. Let  $M_k(\Gamma)$  denote the space of holomorphic modular forms on  $\Gamma$ . If  $\Gamma$  corresponds to an element g in the Monster,  $M_k(\Gamma)$  is also denoted by  $M_k(g)$ .

The Poincaré series of the ring of modular forms on  $\Gamma$  is defined by the following generating function:

$$P_{\Gamma}(u) = 1 + \sum_{k>0, even} d_k(\Gamma)u^k,$$

When  $\Gamma$  is a triangle group, the associated Poincaré series is easily calculated.

-	1	T 5 ( )	
Г		$P_{\Gamma}(u)$	
1A	2,3,∞	$\frac{1}{(1-u^4)(1-u^6)}$	$\Gamma_0(1)$
2A	2,4,∞	$\frac{1+u^6}{(1-u^4)(1-u^8)}$	Γ*(2)
2B	$2,\infty,\infty$	$\frac{1}{(1-u^2)(1-u^4)}$	$\Gamma_0(2)$
3B	$3,\infty,\infty$	$\frac{1+u^4}{(1-u^2)(1-u^6)}$	$\Gamma_0(3)$
4C	$\infty,\infty,\infty$	$\frac{1}{(1-u^2)(1-u^2)}$	$\Gamma_0(4)$
2a	3,3,∞	$\frac{1+u^6}{(1-u^4)(1-u^6)}$	
4a	4,4,∞	$\frac{1 + 2u^6 + u^8}{(1 - u^4)(1 - u^8)}$	

Г		$P_{\Gamma}(u)$	
3A	2,6,∞	$\frac{1 + u^4 + u^6 + 2u^8 + 2u^{10} + u^{12} + 2u^{14} + u^{16} + u^{18}}{(1 - u^{12})(1 - u^{12})}$	Γ*(3)
6a	6,6,∞	$\frac{1 + u^4 + 2u^6 + 3u^8 + 4u^{10} + 3u^{12} + 4u^{14} + 3u^{16} + 2u^{18} + u^{20}}{(1 - u^{12})(1 - u^{12})}$	

#### 3 Modular Forms on 2a

We denote by  $M_k(2a)$  the space of holomorphic modular forms on 2a. Since the group 1A is generated by the group 2a and  $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ , we can decompose  $M_k(2a)$ :

$$M_k(2a) = M_k^+(2a) \oplus M_k^-(2a).$$

where

$$M_k^+(2a) = \{ f(\tau) \in M_k(2a) | f(\tau+1) = f(\tau) \},$$
  
$$M_k^-(2a) = \{ f(\tau) \in M_k(2a) | f(\tau+1) = -f(\tau) \}.$$

Then it is obvious that  $M_k^+(2a) = M_k(1A)$ .

We shall prove that

PROPOSITION 3.1 Let  $\Phi_2$  be the map defined in §1. Then

$$\frac{1}{t_{2a}}F(\frac{1}{6},\frac{1}{2};1;\frac{48\sqrt{-3}}{t_{2a}})^6 = \frac{1}{4}\Phi_2(x^2y^2(x^4-y^4)^2),$$

and this modular form belongs to  $M_6^-(2a)$ .

PROOF.

Put  $T_{2a} = t_{2a} - 24\sqrt{-3}$ . Then  $j(\tau) - 1728 = T_{2a}^2$ .

From Theorem 1.3, it follows that

$$j(\tau) = 16 \frac{(\theta_2(2\tau)^8 + 14\theta_2(2\tau)^4\theta_3(2\tau)^4 + \theta_3(2\tau)^8)^3}{\theta_2(2\tau)^4\theta_3(2\tau)^4(\theta_2(2\tau)^4 - \theta_3(2\tau)^4)^4},$$

and

$$j(\tau) - 1728 = 16 \frac{(\theta_2(2\tau)^{12} - 33\theta_2(2\tau)^8\theta_3(2\tau)^4 - 33\theta_2(2\tau)^4\theta_3(2\tau)^8 + \theta_3(2\tau)^{12})^2}{\theta_2(2\tau)^4\theta_3(2\tau)^4(\theta_2(2\tau)^4 - \theta_3(2\tau)^4)^4}.$$

Hence

$$T_{2a} = \frac{4(\theta_2(2\tau)^{12} - 33\theta_2(2\tau)^8\theta_3(2\tau)^4 - 33\theta_2(2\tau)^4\theta_3(2\tau)^8 + \theta_3(2\tau)^{12})}{\theta_2(2\tau)^2\theta_3(2\tau)^2(\theta_2(2\tau)^4 - \theta_3(2\tau)^4)^2},$$

SO

$$t_{2a} = \frac{4(\theta_2(2\tau)^4 + 2\sqrt{-3}\theta_2(2\tau)^2\theta_3(2\tau)^2 + \theta_3(2\tau)^4)^3}{\theta_2(2\tau)^2\theta_3(2\tau)^2(\theta_2(2\tau)^4 - \theta_3(2\tau)^4)^2}.$$

Thus

$$1 - \frac{48\sqrt{-3}}{t_{2a}} = \frac{(\theta_2(2\tau)^4 - 2\sqrt{-3}\theta_2(2\tau)^2\theta_3(2\tau)^2 + \theta_3(2\tau)^4)^3}{(\theta_2(2\tau)^4 + 2\sqrt{-3}\theta_2(2\tau)^2\theta_3(2\tau)^2 + \theta_3(2\tau)^4)^3},$$

and

$$(1 - \frac{48\sqrt{-3}}{t_{2a}})^{1/3} = \frac{(\theta_2(2\tau)^4 - 2\sqrt{-3}\theta_2(2\tau)^2\theta_3(2\tau)^2 + \theta_3(2\tau)^4)}{(\theta_2(2\tau)^4 + 2\sqrt{-3}\theta_2(2\tau)^2\theta_3(2\tau)^2 + \theta_3(2\tau)^4)}.$$

Put

$$F_{2a} = F(\frac{1}{6}, \frac{1}{2}; 1; \frac{48\sqrt{-3}}{t_{2a}}).$$

From (5) in Proposition 2.1 and Theorem 1.3, it follows that

$$\left(1 - \frac{48\sqrt{-3}}{t_{2a}}\right)^{\frac{1}{3}} F_{2a}^{4} = E_{4}(\tau)$$

$$= \theta_{2}(2\tau)^{8} + 14\theta_{2}(2\tau)^{4}\theta_{3}(2\tau)^{4} + \theta_{3}(2\tau)^{8}$$

Since 
$$x^8 + 14x^4y^4 + y^8 = (x^4 + 2\sqrt{-3}x^2y^2 + y^4)(x^4 - 2\sqrt{-3}x^2y^2 + y^4)$$
, we have 
$$F_{2a}^4 = (\theta_2(2\tau)^4 + 2\sqrt{-3}\theta_2(2\tau)^2\theta_3(2\tau)^2 + \theta_3(2\tau)^4)^2.$$

Hence

$$F_{2a}^{6} = \left\{ \theta_{2}(2\tau)^{4} + 2\sqrt{-3}\theta_{2}(2\tau)^{2}\theta_{3}(2\tau)^{2} + \theta_{3}(2\tau)^{4} \right\}^{3}$$

$$\frac{1}{t_{2a}}F_{2a}^{6} = \frac{1}{4} \left\{ \theta_{2}(2\tau)^{10}\theta_{3}(2\tau)^{2} - 2\theta_{2}(2\tau)^{6}\theta_{3}(2\tau)^{6} + \theta_{2}(2\tau)^{2}\theta_{3}(2\tau)^{10} \right\}.$$

Therefore  $F_{2a}^6$  and  $F_{2a}^6/t_{2a}$  are proved to be holomorphic in the upper half plane and at all cusps.

From (5) in Proposition 2.1 and Theorem 1.3, it follows that

$$\left(1 - \frac{24\sqrt{-3}}{t_{2a}}\right)F_{2a}^6 = E_6(\tau).$$

Hence  $F_{2a}^6$  are proved to be a modular form of weight 6 on 2a. Since the dimension of  $M_6^-(2a)$  is one and  $F_{2a}^6/t_{2a}$  has the Fourier expansion  $q^{1/2}(1-12q+\cdots), F_{2a}^6/t_{2a}$  belongs to  $M_6^-(2a)$ .

This completes the proof of Proposition 3.1.

Theorem 3.2  $\Phi_2$  gives the following isomorphism :

$$\bigoplus_{k\geq 0} M_{2k}^{-}(2a) \cong x^2 y^2 (x^4 - y^4)^2 C[X, Y],$$

where 
$$X = X = x^8 + 14x^4y^4 + y^8, Y = x^{12} - 33x^8y^4 - 33x^4y^8 + y^{12}$$
.

PROOF. It is known that  $\Phi_2$  gives the isomorphism from C[X,Y] to  $\bigoplus_{k\geq 0} M_{2k}(1A) \cong \bigoplus_{k\geq 0} M_{2k}^+(2a)$ . It is obvious that if  $f\in M_{2k}^+(2a)$   $g\in M_{2k'}^-(2a)$ , then  $fg\in M_{2k+2k'}^-(2a)$ . Since the Poincaré series  $P_{2a}(u)=(1+u^6)/(1-u^4)(1-u^6)$ ,.

## 4 Modular Forms on 2B and 2A

The ring of modular forms on 2B was already studied by Maher [14]. We recall his result. He did not use the expression by hypergeometric series. This expression was appeared in the paper by J.Borwein and P.Borwein [2]. So we mix their results.

Put

$$F_{2B} = F(\frac{1}{4}, \frac{3}{4}; 1; \frac{64}{t_{2B}}),$$

and

$$F_{2A} = F(\frac{1}{8}, \frac{3}{8}; 1; \frac{256}{t_{2A}}).$$

By computing a few Fourier coefficients, we have

$$\theta_3(2\tau)^4 + \theta_2(2\tau)^4 = 2E_2(2\tau) - E_2(\tau),$$

where  $E_2(\tau) = 1 - 24 \sum_{n=1}^{\infty} \sigma_1(n) q^n$ . It is well known that  $2E_2(2\tau) - E_2(\tau)$  is a holomorphic modular form on 2B. Hence, from Theorem 2.4, it follows that  $F_{2B}^2$  belongs to  $M_2(2B)$ . From (8) and (11) in Theorem 2.3, it follows that

$$\frac{1}{t_{2B}}F_{2B}^4 = \frac{1}{16}\theta_3(2\tau)^4\theta_2(2\tau)^4.$$

Therefore  $\frac{1}{t_{2B}}F_{2B}^4$  belongs to  $M_4(2B)$ . Since the dimension of  $M_4(2B)$  is two,  $M_4(2B)$  is generated by  $F_{2B}^4$  and  $\frac{1}{t_{2B}}F_{2B}^4$ .

THEOREM 4.1 (Maher)  $\Phi_2$  gives the following isomorphism :

$$\bigoplus_{k>0} M_{2k}(2B) \cong \mathbb{C}[x^4 + y^4, x^4y^4].$$

PROOF. Since the Poincaré series  $P_{2B}(u) = 1/(1-u^4)(1-u^6)$ , this is obvious. From (8) and the algebraic relationship between  $t_{2A}$  and  $t_{2B}$ , it holds

(14) 
$$t_{2A} = \frac{(\theta_3(2\tau)^4 + \theta_2(2\tau)^4)^4}{\theta_3(2\tau)^4\theta_2(2\tau)^4(\theta_3(2\tau)^4 - \theta_2(2\tau)^4)^2}.$$

Hence we have

$$(1 - \frac{256}{t_{2A}})^{1/2} = \frac{\theta_3(2\tau)^8 - 6\theta_3(2\tau)^4\theta_2(2\tau)^4 + \theta_2(2\tau)^8}{(\theta_3(2\tau)^4 + \theta_2(2\tau)^4)^4}.$$

Hence  $(1-\frac{256}{t_{2A}})^{1/2}F_{2A}^6 = \Phi_2((x^8-6x^4y^4+y^8)(x^4+y^4))$  belongs to  $M_6(2B)$ . By computing Fourier expansion of this modular form, we see

$$(1 - \frac{256}{t_{2A}})^{1/2} F_{2A}^6 = \frac{1}{9} \left\{ E_6(\tau) + 8E_6(2\tau) \right\}.$$

This shows that  $(1 - \frac{256}{t_{2A}})^{1/2} F_{2A}^6$  belongs to  $M_6(2A)$ .

THEOREM 4.2  $\Phi_2$  gives the following isomorphisms:

$$\bigoplus_{k\geq 0} M_{4k}(2A) \cong \mathbb{C}[(x^4 + y^4)^2, x^4y^4(x^4 - y^4)^2].$$

$$\bigoplus_{k>0} M_{4k+2}(2A) \cong (x^8 - 6x^4y^4 + y^8)(x^4 + y^4)C[(x^4 + y^4)^2, x^4y^4(x^4 - y^4)^2].$$

#### 5 Modular Forms on 4a

Put

$$F_{4a} = F(\frac{1}{4}, \frac{1}{2}; 1; \frac{32\sqrt{-1}}{t_{4a}}),$$

We denote by  $M_k(4a)$  the space of holomorphic modular forms on 4a. Since the group 2A is generated by the group 4a and  $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ , we can decompose  $M_k(4a)$ :

$$M_k(4a) = M_k^+(4a) \oplus M_k^-(4a).$$

where

$$M_k^+(4a) = \{ f(\tau) \in M_k(4a) | f(\tau+1) = f(\tau) \},$$
  
$$M_k^-(4a) = \{ f(\tau) \in M_k(4a) | f(\tau+1) = -f(\tau) \}.$$

Then it is obvious that

$$M_k^+(4a) = M_k(2A).$$

PROPOSITION 5.1 Let  $\Phi_2$  be the map defined in §1. Then

$$\frac{1}{t_{4a}} \left( 1 - \frac{16\sqrt{-1}}{t_{4a}} \right)^{\frac{1}{4}} F_{4a}^{6} = \frac{1}{2} \Phi_2(x^2 y^2 (x^8 - y^8)),$$

and this modular form belongs to  $M_6^-(4a)$ .

PROOF.

Put 
$$T_{4a} = t_{4a} - 16\sqrt{-1}$$
. Then  $t_{2A} - 256 = T_{4a}^2$ .

By (14) in §4, we have

(15) 
$$T_{4a} = \frac{4(\theta_2(2\tau)^8 - 6\theta_2(2\tau)^4\theta_3(2\tau)^4 + \theta_3(2\tau)^8)}{\theta_2(2\tau)^2\theta_3(2\tau)^2(\theta_2(2\tau)^4 - \theta_3(2\tau)^4)},$$

SO

(16) 
$$t_{4a} = \frac{4(\theta_3(2\tau)^2 + \sqrt{-1}\theta_2(2\tau)^2)^4}{\theta_2(2\tau)^2\theta_3(2\tau)^2(\theta_2(2\tau)^4 - \theta_3(2\tau)^4)}.$$

Thus

$$1 - \frac{32\sqrt{-1}}{t_{4a}} = \frac{4(\theta_3(2\tau)^2 - \sqrt{-1}\theta_2(2\tau)^2)^4}{4(\theta_3(2\tau)^2 + \sqrt{-1}\theta_2(2\tau)^2)^4},$$

SO

(17) 
$$\left(1 - \frac{32\sqrt{-1}}{t_{4a}}\right)^{1/2} = \frac{(\theta_3(2\tau)^2 - \sqrt{-1}\theta_2(2\tau)^2)^2}{(\theta_3(2\tau)^2 + \sqrt{-1}\theta_2(2\tau)^2)^2}.$$

By (6) in Proposition 2.1, we have

$$\left(1 - \frac{32\sqrt{-1}}{t_{4a}}\right)^{\frac{1}{2}} F_{4a}^{4} = F_{2A}^{4}$$

$$= \Phi_{2}(x^{8} + 2x^{4}y^{4} + x^{8}).$$

Substituting (17) in this identity, we have

(18) 
$$F_{4a}^4 = \Phi_2((x^2 + \sqrt{-1}y^2)^4).$$

From (16),(17),and (18), it follows

(19) 
$$\frac{1}{t_{4a}} \left( 1 - \frac{32\sqrt{-1}}{t_{4a}} \right)^{\frac{1}{4}} F_{4a}^{6} = \frac{1}{2} \Phi_{2}(x^{2}y^{2}(x^{8} - y^{8}))$$

We see

(20) 
$$\left(1 - \frac{256}{t_{2A}}\right)^{\frac{1}{2}} F_{2A}^{6} = \frac{T_{4a}}{t_{4a}} \left(1 - \frac{32\sqrt{-1}}{t_{4a}}\right)^{\frac{1}{4}} F_{4a}^{6}.$$

Hence the right hand side of (20) is a modular form of weight 6 on 4a, and (19) shows that this is a holomorphic modular form on 4a.

PROPOSITION 5.2 Let  $\Phi_2$  be the map defined in §1. Then

(21) 
$$\frac{1}{t_{4a}} \left( 1 - \frac{16\sqrt{-1}}{t_{4a}} \right) F_{4a}^{8} = \frac{1}{4} \Phi_{2}(x^{2}y^{2}(x^{4} - y^{4})((x^{8} - 6x^{4}y^{4} + y^{8})).$$

This modular form belongs to  $M_8^-(4a)$ .

PROOF. Since  $\left(1 - \frac{32\sqrt{-1}}{t_{4a}}\right)^{\frac{1}{2}} F_{4a}^4$  belongs to  $M_4(4a)$ , the left hand side of (21) is a modular form of weight 8 on 4a. Since the left hand side of (21) is equal to  $\frac{1}{t_{4a}} \frac{T_{4a}}{t_{4a}} F_{4a}^8$ , the identity (21) follows from (15),(16), and (18). Since the dimension of  $M_8^-(4a)$  is 1 and the left hand side of (21) has the Fourier expansion  $q^{1/2}(1 - 84q - 82q^2 + \cdots)$ , we obtain the proof.

PROPOSITION 5.3 Let  $\Phi_2$  be the map defined in §1. Then

(22) 
$$\frac{1}{t_{4a}^2} F_{4a}^8 = \frac{1}{16} \Phi_2 (x^4 y^4 (x^4 - y^4)^2).$$

This modular form belongs to  $M_8^+(4a)$ .

PROOF.

We see that  $\frac{1}{t_{2A}}F_{2A}^8 = \frac{1}{t_{4a}^2}F_{4a}^8$ . So we obtain the proof. Combining Proposition 5.2 with 5.3, we obtain

Theorem 5.4  $\Phi_2$  gives the following isomorphisms:

$$\bigoplus_{k\geq 0} M_{4k}^{-}(4a) \cong \left\{ x^2 y^2 (x^4 - y^4) (x^8 - 6x^4 y^4 + y^8) \right\} C[(x^4 + y^4)^2, x^4 y^4 (x^4 - y^4)^2].$$

$$\bigoplus_{k\geq 0} M_{4k+2}^{-}(4a) \cong \left\{ x^2 y^2 (x^4 - y^4) (x^4 + y^4) \right\} C[(x^4 + y^4)^2, x^4 y^4 (x^4 - y^4)^2],$$

#### 6 Modular forms on 4C

The ring of holomorphic modular forms on 2B was already studied by Maher [14]. But he did not use the expression in terms of hypergeometric functions. This expression was appeared in the paper by J.Borwein and P.Borwein [2]. So we mix their results.

Put

$$F_{4C} = F(\frac{1}{2}, \frac{1}{2}; 1; \frac{16}{t_{4C}}).$$

From (12) in Theorem 2.3 and (9), it follows that

$$F_{4C}^2 = \theta_3 (2\tau)^4, \frac{1}{t_{4C}} F_{4C}^2 = \frac{1}{16} \theta_2 (2\tau)^4.$$

Theorem 6.1 (Maher)  $\Phi_2$  gives the following isomorphism :

$$\bigoplus_{k>0} M_{2k}(4C) \cong \mathbf{C}[x^2, y^2].$$

PROOF. Since the Poincaré series  $P_{4C}(u) = 1/(1-u^2)(1-u^2)$ , this is obvious.

## 7 Modular forms on 3B and 3A

The ring of holomorphic modular forms on 3B was already studied by Maher [14]. The situation is the same as in the case of 4C. But the structure of the ring of holomorphic modular forms on 3A was not studied.

Let  $\Phi_3: \mathbf{C}[x,y] \to \mathbf{C}[[q]]$  be the map defined in §1 by

$$\Phi_3(f(x,y)) = f(\psi_0(\tau), \psi_1(\tau)).$$

Put

$$F_{3B} = F(\frac{1}{3}, \frac{2}{3}; 1; \frac{27}{t_{3B}}).$$

From Theorem 2.3, we have  $F_{3B}^2 = \Phi_3(x^2)$ . Combining this with (10), we have

$$\frac{1}{t_{3B}}F_{3B}^4 = \frac{1}{27}\Phi_3(xy^3),$$

and

$$\frac{1}{t_{3B}^2}F_{3B}^6 = \frac{1}{729}\Phi_3(y^6).$$

Theorem 7.1  $\Phi_3$  gives the following isomorphism :

$$\bigoplus_{k>0} M_{2k}(3B) \cong \mathbf{C}[x^2, y^6] \oplus xy^3 \mathbf{C}[x^2, y^6].$$

PROOF. Since the Poincaré series  $P_{3B}(u) = (1+u^4)/(1-u^2)(1-u^6)$ , this is obvious. From Table 3, it follows that

(23) 
$$t_{3A} = 27 \frac{\psi_0^6}{\psi_1^3(\psi_0^3 - \psi_1^3)},$$

and

(24) 
$$(1 - \frac{108}{t_{3A}})^{\frac{1}{2}} = 1 - 2\frac{27}{t_{3B}} = \frac{\psi_0^3 - 2\psi_1^3}{\psi_0^3}.$$

Put

$$F_{3A} = F(\frac{1}{6}, \frac{1}{3}; 1; \frac{108}{t_{3A}}).$$

From Proposition 2.2, it follows  $F_{3A} = F_{3B}$ .

By computing a few Fourier coefficients, we see

$$F_{3A}^4 = \frac{1}{10} \left\{ E_4(\tau) + 9E_4(3\tau) \right\},\,$$

which shows that  $F_{3A}^4 \in M_4(3A)$ .

From (22), it follows that

(25) 
$$(1 - \frac{108}{t_{3A}})^{\frac{1}{2}} F_{3A}^6 = \psi_0^3 (\psi_0^3 - 2\psi_1^3).$$

This belongs to  $M_6(3B)$ . By computing a few Fourier coefficients, we see that this is equal to  $\frac{1}{28} \{ E_6(\tau) + 27E_6(3\tau) \}$ , which shows that this modular form belongs to  $M_6(3A)$ .

Since  $F_{3A}^4 \in M_4(3A)$ ,  $F_{3A}^8 \in M_8(3A)$ . To show that  $\frac{1}{t_{3A}}F_{3A}^8 \in M_8(3A)$ , it is enough to show that it is holomorphic on the upper half plane and cusps. this follows from the following identity:

$$\frac{1}{t_{3A}}F_{3A}^{8} = \frac{1}{27}\psi_0^2\psi_1^3(\psi_0^3 - \psi_1^3).$$

By using the same argument, we can prove the following identities:

$$(1 - \frac{108}{t_{3A}})^{\frac{1}{2}} F_{3A}^{10} = \psi_0^7 (\psi_0^3 - 2\psi_1^3),$$

$$\frac{1}{t_{3A}} (1 - \frac{108}{t_{3A}})^{\frac{1}{2}} F_{3A}^{10} = \psi_0^3 \psi_1^3 (\psi_0^3 - 2\psi_1^3) (\psi_0^3 - \psi_1^3).$$

$$\frac{1}{t_{3A}}F_{3A}^{12} = \frac{1}{27}\psi_0^6\psi_1^3(\psi_0^3 - \psi_1^3),$$

$$\frac{1}{t_{3A}^2}F_{3A}^{12} = \frac{1}{729}\psi_1^6(\psi_0^3 - \psi_1^3)^2.$$

$$(1 - \frac{108}{t_{3A}})^{\frac{1}{2}} F_{3A}^{14} = \psi_0^{11} (\psi_0^3 - 2\psi_1^3),$$

$$\frac{1}{t_{3A}} (1 - \frac{108}{t_{3A}})^{\frac{1}{2}} F_{3A}^{14} = \psi_0^5 \psi_1^3 (\psi_0^3 - 2\psi_1^3) (\psi_0^3 - \psi_1^3).$$

$$\frac{1}{t_{3A}}F_{3A}^{16} = \frac{1}{27}\psi_0^{10}\psi_1^3(\psi_0^3 - \psi_1^3),$$

$$\frac{1}{t_{3A}^2}F_{3A}^{16} = \frac{1}{729}\psi_0^4\psi_1^6(\psi_0^3 - \psi_1^3)^2.$$

$$(1 - \frac{108}{t_{3A}})^{\frac{1}{2}} F_{3A}^{18} = \psi_0^{15} (\psi_0^3 - 2\psi_1^3),$$

$$\frac{1}{t_{3A}} (1 - \frac{108}{t_{3A}})^{\frac{1}{2}} F_{3A}^{18} = \psi_0^9 \psi_1^3 (\psi_0^3 - 2\psi_1^3) (\psi_0^3 - \psi_1^3),$$

$$\frac{1}{t_{3A}^2} (1 - \frac{108}{t_{3A}})^{\frac{1}{2}} F_{3A}^{18} = \psi_0^3 \psi_1^6 (\psi_0^3 - 2\psi_1^3) (\psi_0^3 - \psi_1^3)^2.$$

Let S denote the subring generated by  $X = x^2$  and  $Y = y^3(x^3 - y^3)$  of C[x, y]. We define the degree of the monomial  $X^aY^b$  is 2a + 6b. Let  $S_0$  and  $S_1$  denote the submodules of S generated by all monomials of X and Y of degree 4k and 4k + 2 for all integers  $k \ge 0$ , respectively. Put

$$\tilde{S}_k = \{(a,b)| 2a + 6b = k, a, b \ge 0, \text{and } a, b \in \mathbf{Z}\}.$$

Let  $d_k$  denote the number of elements in the set  $\tilde{S}_k$ . Then it is proved that  $d_{4k-2} = \dim M_{4k+2}(3A)$ .

Combining this with the above identities and the Poincaré series of 3A in §2.1, we obtain the following theorem.

THEOREM 7.2  $\Phi_3$  gives the following isomorphisms:

$$\bigoplus_{k\geq 0} M_{4k}(3A) \cong S_0$$
  
 $\bigoplus_{k\geq 0} M_{4k+2}(3A) \cong \left\{ x(x^3 - 2y^3) \right\} S_1.$ 

REMARK. If we compare the above theorem with Theorem 1.6, we notice the similarity.

#### 8 Modular forms on 6a

The connection between the ring of modular forms on 3B and 3A and a subring of the polynomial ring of two variables x and y are expressed in terms of  $\Phi_3$  which is introduced in studying theta functions associated to ternary code. However, to study the connection between the ring of modular forms on 6a and a subring of the polynomial ring of two variables x and y, we need a new map  $\Psi_4$  which is related to the map  $\Phi_4$  introduced in studying theta functions associated to quaternary codes.

We define the map  $\Psi_4: \mathbf{C}[x,y] \to \mathbf{C}[[q]]$  by

$$= \Psi_4(f(x,y)) = f(\phi_0(2\tau), \phi_1(2\tau)).$$

So  $\Psi_4$  is different from the map  $\Phi_4$  defined in §1.

PROPOSITION 8.1 The following identities of theta functions hold:

(26) 
$$\phi_0(2\tau)^2 + 3\phi_1(2\tau)^2 = \phi_0(\tau)^2 = \psi_0(\tau)^2,$$

(27) 
$$\frac{1}{4}\phi_1(2\tau)^2(\phi_0(2\tau)^2 - \phi_1(2\tau)^2)^2 = \frac{1}{27}\psi_1(\tau)^3(\psi_0(\tau)^3 - \psi_1(\tau)^3)$$

PROOF.

The identity (24) was given in the page 176 in Sloane [18]. In [18], the left side of (25) is shown to be equal to  $\eta(\tau)^6\eta(3\tau)^6$ , which is known to be an element in  $M_6(3B)$ . Using Theorem 7.1, we can prove (25).

Let S be the ring defined in the precedin section, which is generated by  $x^2$  and  $y^3(x^3 - y^3)$ . Then  $\Phi_3(S)$  is a subring in C[[q]].

COROLLARY 8.2  $\Psi_4$  gives the following isomorphism:

$$C[x^2 + 3y^2, y^2(x^2 - y^2)^2] \cong \Phi_3(T)$$

Let T denote the subring generated by  $U = x^2 + 3y^2$  and  $V = y^2(x^2 - y^2)^2$  of C[x, y]. We define the degree of the monomial  $U^aV^b$  is 2a + 6b. Let  $T_0$  and  $T_1$  denote the submodules of T generated by all monomials of U and V of degree 4k and 4k + 2 for all integers  $k \geq 0$ , respectively.

We denote by  $M_k(6a)$  the space of holomorphic modular forms on 6a. By the same reason as in the case of 2a and 4a, we can decompose  $M_k(6a)$ :

$$M_k^+(6a) = \{ f(\tau) \in M_k(6a) | f(\tau+1) = f(\tau) \},$$
  
 $M_k^-(6a) = \{ f(\tau) \in M_k(6a) | f(\tau+1) = -f(\tau) \}.$ 

Then

$$M_k(6a) = M_k^+(6a) \oplus M_k^-(6a),$$

and we have

$$M_k^+(6a) = M_k(3A).$$

We express holomorphic modular forms on 6a in terms of  $F_{6a}$  and  $t_{6a}$ . Put  $T_{6a} = t_{6a} - 6\sqrt{-3}$ . Then  $t_{3A} - 108 = T_{6a}^2$  and  $t_{3A} = t_{6a}(t_{6a} - 12\sqrt{-3})$ . From (23) and Proposition 8.1, we have

$$t_{3A} = \frac{4(\phi_0(2\tau)^2 + 3\phi_1(2\tau)^2)^3}{\phi_1(2\tau)^2(\phi_0(2\tau)^2 - \phi_1(2\tau)^2)^2}.$$

$$t_{3A} - 108 = \left\{ \frac{2\phi_0(2\tau)(\phi_0(2\tau)^2 - 9\phi_1(2\tau)^2)}{\phi_1(2\tau)(\phi_0(2\tau)^2 - \phi_1(2\tau)^2)} \right\}^2,$$

SO

$$T_{6a} = \frac{2\phi_0(2\tau)(\phi_0(2\tau)^2 - 9\phi_1(2\tau)^2)}{\phi_1(2\tau)(\phi_0(2\tau)^2 - \phi_1(2\tau)^2)}.$$

Thus

(28) 
$$t_{6a} = \frac{2(\phi_0(2\tau) + \sqrt{-3}\phi_1(2\tau))^3}{\phi_1(2\tau)(\phi_0(2\tau)^2 - \phi_1(2\tau)^2)},$$

and

(29) 
$$1 - \frac{12\sqrt{-3}}{t_{6a}} = \frac{(\phi_0(2\tau) - \sqrt{-3}\phi_1(2\tau))^3}{(\phi_0(2\tau) + \sqrt{-3}\phi_1(2\tau))^3},$$

$$1 - \frac{6\sqrt{-3}}{t_{6a}} = \frac{\phi_0(2\tau)(\phi_0(2\tau)^2 - 9\phi_1(2\tau)^2)}{(\phi_0(2\tau) + \sqrt{-3}\phi_1(2\tau)^3)}.$$

Hence

(30) 
$$\left(1 - \frac{12\sqrt{-3}}{t_{6a}}\right)^{1/3} = \frac{\phi_0(2\tau) - \sqrt{-3}\phi_1(2\tau)}{\phi_0(2\tau) + \sqrt{-3}\phi_1(2\tau)}.$$

Put

$$F_{6a} = F(\frac{1}{3}, \frac{1}{2}; 1; \frac{12\sqrt{-3}}{t_{3A}}).$$

From (7) in Proposition 2.1, it follows that  $F_{3A} = \left(1 - \frac{12\sqrt{-3}}{t_{6a}}\right)^{\frac{1}{6}} F_{6a}$ .  $F_{3A}^4$  belongs to  $M_4(3A)$  and  $F_{3A}^4 = \phi_0(\tau)^4 = \Psi_4((x^2 + 3y^2)^2)$ . From these and Proposition 8.1 it follows that

$$\left(1 - \frac{12\sqrt{-3}}{t_{6a}}\right)^{\frac{2}{3}} F_{6a}^{4} = (\phi_0(2\tau)^2 + 3\phi_1(2\tau)^2)^2.$$

Combining this with (30), we have

(31) 
$$F_{6a} = \Psi_4(x + \sqrt{-3}y).$$

Similarly, we have

$$\left(1 - \frac{108}{t_{3A}}\right)^{\frac{1}{2}} F_{3A}^{6} = \left(1 - \frac{12\sqrt{-3}}{t_{6a}}\right)^{\frac{1}{2}} \left(1 - \frac{6\sqrt{-3}}{t_{6a}}\right) F_{6a}^{6}.$$

From (28),(29) and (31), we have

$$\frac{1}{t_{6a}} \left( 1 - \frac{12\sqrt{-3}}{t_{6a}} \right)^{\frac{1}{2}} F_{6a}^{6} = \frac{1}{2} \phi_0(\tau) \Psi_4(y(x^2 + 3y^2)(x^2 - y^2)).$$

Here we use the strange fact that  $\phi_0(\tau) = \{\phi_0(2\tau)^2 + 3\phi_1(2\tau)^2\}^{1/2}$  which follows from Proposition 8.1. Therefore  $\frac{1}{t_{6a}} \left(1 - \frac{12\sqrt{-3}}{t_{6a}}\right)^{\frac{1}{2}} F_{6a}^6$  belongs to  $M_6(6a)$ . Since the dimension of  $M_6^-(6a)$  is 1, this modular form is proved to be an element of  $M_6^-(6a)$  by computing a few Fourier coefficients. Similarly we obtain

$$\left(1 - \frac{6\sqrt{-3}}{t_{6a}}\right) \left(1 - \frac{12\sqrt{-3}}{t_{6a}}\right)^{\frac{1}{2}} F_{6a}^{6} = \phi_0(\tau) \Psi_4(x(x^2 + 3y^2)(x^2 - 9y^2)).$$

This modular form belongs to  $M_6^+(6a)$ .

By the same argument as above, we can express holomorphic modular forms on 6a in terms of  $t_{6a}$  and  $F_{6a}$ . We give only identities without proof:

$$\left(1 - \frac{12\sqrt{-3}}{t_{6a}}\right)^{\frac{4}{3}} F_{6a}^{8} = \Psi_{4}((x^{2} + 3y^{2})^{4}).$$

$$\frac{1}{t_{6a}^{2}} \left(1 - \frac{12\sqrt{-3}}{t_{6a}}\right)^{\frac{1}{3}} F_{6a}^{8} = \frac{1}{4}\Psi_{4}(y^{2}(x^{2} - y^{2})^{2}(x^{2} + 3y^{2})).$$

$$\frac{1}{t_{6a}} \left(1 - \frac{6\sqrt{-3}}{t_{6a}}\right) \left(1 - \frac{12\sqrt{-3}}{t_{6a}}\right)^{\frac{1}{3}} F_{6a}^{8} = \frac{1}{2}\Psi_{4}(y(x^{2} - y^{2})x(x^{2} - 9y^{2})(x^{2} + 3y^{2})).$$

$$\left(1 - \frac{6\sqrt{-3}}{t_{6a}}\right) \left(1 - \frac{12\sqrt{-3}}{t_{6a}}\right)^{\frac{7}{6}} F_{6a}^{10} = \phi_{0}(\tau)\Psi_{4}(x(x^{2} - 9y^{2})(x^{2} + 3y^{2})^{3}).$$

$$\frac{1}{t_{6a}^{2}} \left(1 - \frac{6\sqrt{-3}}{t_{6a}}\right) \left(1 - \frac{12\sqrt{-3}}{t_{6a}}\right)^{\frac{7}{6}} F_{6a}^{10} = \frac{1}{4}\phi_{0}(\tau)\Psi_{4}(y^{2}(x^{2} - y^{2})^{2}x(x^{2} - 9y^{2})).$$

$$\frac{1}{t_{6a}^{3}} \left(1 - \frac{12\sqrt{-3}}{t_{6a}}\right)^{\frac{7}{6}} F_{6a}^{10} = \frac{1}{2}\phi_{0}(\tau)\Psi_{4}(y(x^{2} - y^{2})(x^{2} + 3y^{2})^{3}).$$

$$\frac{1}{t_{6a}^{3}} \left(1 - \frac{12\sqrt{-3}}{t_{6a}}\right)^{2} F_{6a}^{12} = \frac{1}{8}\phi_{0}(\tau)\Psi_{4}(y^{3}(x^{2} - y^{2})^{3}).$$

$$\left(1 - \frac{12\sqrt{-3}}{t_{6a}}\right)^{2} F_{6a}^{12} = \frac{1}{4}\Psi_{4}(y^{2}(x^{2} + 3y^{2})^{6}).$$

$$\frac{1}{t_{6a}^{2}} \left(1 - \frac{12\sqrt{-3}}{t_{6a}}\right) F_{6a}^{12} = \frac{1}{4}\Psi_{4}(y^{2}(x^{2} - y^{2})^{2}(x^{2} + 3y^{2})^{3}).$$

$$\frac{1}{t_{6a}^{4}} F_{6a}^{12} = \frac{1}{16}\Psi_{4}(y^{4}(x^{2} - y^{2})^{4}).$$

The denominator of the Poincaré series of 6a is  $(1 - u^{12})^2$ . The two generators are modular forms of weight 12 given by

$$\left(1 - \frac{12\sqrt{-3}}{t_{6a}}\right)^2 F_{6a}^{12}, \frac{1}{t_{6a}^4} F_{6a}^{12}.$$

Let  $d_k$  denote the number of elements in the set  $\tilde{S}_k$ . Then it is proved that

$$d_{4k-2} = \dim M_{4k+2}^{-}(6a) = \dim M_{4k+4}^{-}(6a).$$

Combinining this with the above identities, we obtain the proof of the following theorem.

Theorem 8.3  $\Psi_4$  gives the following isomorphisms:

$$\bigoplus_{k\geq 0} M_{4k}^+(6a) \cong S_0,$$

$$\phi_0(\tau)^{-1} \bigoplus_{k\geq 0} M_{4k+2}^+(6a) \cong x(x^2 - 9y^2) S_1.$$

$$\bigoplus_{k\geq 0} M_{4k}^-(6a) \cong x(x^2 - 9y^2) y(x^2 - y^2) S_1.$$

$$\phi_0(\tau)^{-1} \bigoplus_{k\geq 0} M_{4k+2}^-(6a) \cong y(x^2 - y^2) S_1,$$

REMARK. In §7 we gave the basis of  $M_k(3A)$  in terms of  $F_{3A}$  and  $t_{3A}$ . It is observed that this basis of  $M_k(3A)$  is the same as the basis of  $M_k^+(6a)$  given in terms of  $F_{6a}$  and  $t_{6a}$ .

### 9 Concluding remarks

Nine non-compact arithmetic triangle groups are divided into three parts according to their properties;

$$\{1A, 2A, 3A\}, \{2B, 3B, 4C\}, \{2a, 4a, 6a\}.$$

It seems that the theory of modular forms on each group in the same set holds many properties in common. We will discuss some properties in subsequent papers under preparation.

#### 10 Table

We give a few Fourier coefficients of generators of the ring of modular forms on nine groups. By Remark at the end of §8, we know that the generators of the ring of modular forms  $\bigoplus_{k\geq 0} M_{2k}^+(6a)$  given in §8 are the same as those of  $\bigoplus_{k\geq 0} M_{2k}(3A)$ , so we omit them from our table.

#### Case of 1A

$$F_{1A}^{4} = E_{4}(\tau),$$

$$= 1 + 240q + 2160q^{2} + 6720q^{3} + \cdots \qquad \text{A4O69}$$

$$(1 - \frac{12^{3}}{j(\tau)})^{\frac{1}{2}}F_{1A}^{6} = E_{6}(\tau)$$

$$= 1 - 504q - 16632q^{2} - 122976q^{3} + \cdots$$

$$\text{A1397-3}$$

Case of 2A

$$F_{2A}^{4} = 1 + 48q + 624q^{2} + 1344q^{3} + 5232q^{4} + \cdots$$

$$= \frac{1}{5} \{ E_{4}(\tau) + 4E_{4}(2\tau) \}$$

$$(1 - \frac{256}{t_{2A}})^{\frac{1}{2}} F_{2A}^{6} = 1 - 56q - 2296q^{2} - 13664q^{3} - 73976q^{4} + \cdots$$

$$= \frac{1}{9} \{ E_{6}(\tau) + 8E_{6}(2\tau) \}$$

$$\frac{1}{t_{2A}} F_{2A}^{8} = q - 8q^{2} + 12q^{3} + 64q^{4} - 210q^{5} + \cdots$$

$$= \eta(2\tau)^{8} \eta(\tau)^{8}$$

$$A = 2288$$

Case of 2B

$$F_{2B}^{2} = 1 + 24q + 24q^{2} + 96q^{3} + 24q^{4} + 144q^{5} + \cdots$$

$$= 2E_{2}(2\tau) - E_{2}(\tau),$$

$$\frac{1}{t_{2B}}F_{2B}^{4} = q + 8q^{2} + 28q^{3} + 64q^{4} + 126q^{5} + \cdots$$

$$A = 4011$$

$$= \frac{\eta(2\tau)^{16}}{\eta(\tau)^8},$$

Case of 3B

$$F_{3B}^{2} = 1 + 12q + 36q^{2} + 12q^{3} + 84q^{4} + 72q^{5} + \cdots$$

$$= -\frac{1}{2} \{E_{2}(\tau) - 3E_{2}(3\tau)\}$$

$$\frac{1}{t_{3B}} F_{3B}^{4} = q + 9q^{2} + 27q^{3} + 73q^{4} + 126q^{5} + 243q^{6} + \cdots$$

$$\frac{1}{t_{3B}^{2}} F_{3B}^{6} = q^{2} + 6q^{3} + 27q^{4} + 80q^{5} + 207q^{6} + \cdots$$

$$= \left(\frac{\eta(3\tau)^{9}}{\eta(\tau)^{3}}\right)^{2}$$

$$A = \left(\frac{\eta(3\tau)^{9}}{\eta(\tau)^{3}}\right)^{2}$$

$$A = \left(\frac{\eta(3\tau)^{9}}{\eta(\tau)^{3}}\right)^{2}$$

Case of 4C

$$F_{4C}^{2} = 1 + 8q + 24q^{2} + 32q^{3} + 24q^{4} + 48q^{5} + \cdots$$

$$= -\frac{1}{3} \{ E_{2}(\tau) - 4E_{2}(4\tau) \}$$

$$\frac{1}{t_{4C}} F_{4C}^{2} = q + 4q^{3} + 6q^{5} + 8q^{7} + \cdots$$

$$= \frac{\eta(4\tau)^{8}}{\eta(2\tau)^{4}}$$

$$\Rightarrow 8438.$$

Case of 2a

$$\left(1 - \frac{48\sqrt{-3}}{t_{2a}}\right)^{\frac{1}{3}} F_{2a}^{4} = 1 + 240q + 2160q^{2} + 6720q^{3} + \cdots 
= E_{4}(\tau)$$

$$\left(1 - \frac{24\sqrt{-3}}{t_{2a}}\right) F_{2a}^{6} = 1 - 504q - 16632q^{2} + \cdots$$

$$\frac{1}{t_{2a}} F_{2a}^{6} = q^{\frac{1}{2}} - 12q^{\frac{3}{2}} + 54q^{\frac{5}{2}} - 88q^{\frac{7}{2}} + \cdots$$

$$A + 35$$

$$\left(1 - \frac{32\sqrt{-1}}{t_{4a}}\right)^{\frac{1}{2}} F_{4a}^{4} = \frac{1}{5} \left\{ E_{4}(\tau) + 4E_{4}(2\tau) \right\}$$

$$\left(1 - \frac{16\sqrt{-1}}{t_{4a}}\right) \left(1 - \frac{32\sqrt{-1}}{t_{4a}}\right)^{\frac{1}{4}} F_{4a}^{6} = 1 - 56q - 2296q^{2} - 13664q^{3} - 73976q^{4} + \cdots \right)$$

$$\left(1 - \frac{32\sqrt{-1}}{t_{4a}}\right)^{\frac{1}{4}} F_{4a}^{6} = q^{\frac{1}{2}} + 20q^{\frac{3}{2}} - 74q^{\frac{5}{2}} - 24q^{\frac{7}{2}} + \cdots \right)$$

$$\left(1 - \frac{32\sqrt{-1}}{t_{4a}}\right)^{\frac{1}{4}} F_{4a}^{6} = q^{\frac{1}{2}} + 20q^{\frac{3}{2}} - 74q^{\frac{5}{2}} - 24q^{\frac{7}{2}} + \cdots \right)$$

$$\left(1 - \frac{32\sqrt{-1}}{t_{4a}}\right)^{\frac{1}{4}} F_{4a}^{6} = q^{\frac{1}{2}} + 20q^{\frac{3}{2}} - 74q^{\frac{5}{2}} - 24q^{\frac{7}{2}} + \cdots \right)$$

$$\left(1 - \frac{32\sqrt{-1}}{t_{4a}}\right)^{\frac{1}{4}} F_{4a}^{8} = q^{\frac{1}{2}} + 84q^{\frac{3}{2}} - 82q^{\frac{5}{2}} - 456q^{\frac{7}{2}} + \cdots \right)$$

$$\left(1 - \frac{32\sqrt{-1}}{t_{4a}}\right)^{\frac{1}{4}} F_{4a}^{8} = q^{\frac{1}{2}} - 84q^{\frac{3}{2}} - 82q^{\frac{5}{2}} - 456q^{\frac{7}{2}} + \cdots \right)$$

$$\left(1 - \frac{32\sqrt{-1}}{t_{4a}}\right)^{\frac{1}{4}} F_{4a}^{8} = q^{-8}q^{2} + 12q^{3} + 64q^{4} - 210q^{5} + \cdots \right)$$

$$\left(1 - \frac{32\sqrt{-1}}{t_{4a}}\right)^{\frac{1}{4}} F_{4a}^{8} = q^{-8}q^{2} + 12q^{3} + 64q^{4} - 210q^{5} + \cdots \right)$$

$$\left(1 - \frac{32\sqrt{-1}}{t_{4a}}\right)^{\frac{1}{4}} F_{4a}^{8} = q^{-8}q^{2} + 12q^{3} + 64q^{4} - 210q^{5} + \cdots \right)$$

$$\left(1 - \frac{32\sqrt{-1}}{t_{4a}}\right)^{\frac{1}{4}} F_{4a}^{8} = q^{-8}q^{2} + 12q^{3} + 64q^{4} - 210q^{5} + \cdots \right)$$

Case of 3A

$$F_{3A}^{4} = 1 + 24q + 216q^{2} + 888q^{3} + 1752q^{4} + \cdots$$

$$= \frac{1}{10} \left\{ E_{4}(\tau) + 9E_{4}(3\tau) \right\}$$

$$\left( 1 - \frac{108}{t_{3A}} \right)^{\frac{1}{2}} F_{3A}^{6} = 1 - 18q - 594q^{2} - 4878q^{3} - 19026q^{4} - \cdots$$

$$= \frac{1}{28} \left\{ E_{6}(\tau) + 27E_{6}(3\tau) \right\}$$

$$F_{3A}^{8} = 1 + 48q + 1008q^{2} + 12144q^{3} + 92784q^{4} + \cdots$$

$$\left( 1 - \frac{108}{t_{3A}} \right)^{\frac{1}{2}} F_{3A}^{6} = q + 6q^{2} - 27q^{3} - 92q^{4} + 390q^{5} + \cdots$$

$$\left( 1 - \frac{108}{t_{3A}} \right)^{\frac{1}{2}} F_{3A}^{10} = 1 + 6q - 810q^{2} - 22134q^{3} - 278634q^{4} - \cdots$$

$$\frac{1}{t_{3A}} \left( 1 - \frac{108}{t_{3A}} \right)^{\frac{1}{2}} F_{3A}^{10} = q - 36q^{2} - 81q^{3} + 784q^{4} - 1314q^{5} + \cdots$$

$$F_{3A}^{12} = 1 + 72q + 2376q^2 + 47592q^3 + 646344q^4 + \cdots$$

$$\frac{1}{t_{3A}}F_{3A}^{12} = q + 30q^2 + 333q^3 + 1444q^4 - 570q^5 + \cdots$$

$$\frac{1}{t_{3A}^2}F_{3A}^{12} = q^2 - 12q^3 + 54q^4 - 100q^5 + 45q^6 + \cdots$$

$$\left(1 - \frac{108}{t_{3A}}\right)^{\frac{1}{2}}F_{3A}^{14} = 1 + 30q - 450q^2 - 39390q^3 - 977730q^4 - \cdots$$

$$\frac{1}{t_{3A}}\left(1 - \frac{108}{t_{3A}}\right)^{\frac{1}{2}}F_{3A}^{14} = q - 12q^2 - 729q^3 - 8048q^4 - 30210q^5 + \cdots$$

$$\frac{1}{t_{3A}}F_{3A}^{16} = q + 54q^2 + 1269q^3 + 16804q^4 + 134406q^5 + \cdots$$

$$\frac{1}{t_{3A}}\left(1 - \frac{108}{t_{3A}}\right)^{\frac{1}{2}}F_{3A}^{18} = q + 12q^2 - 801q^3 - 27248q^4 - 389730q^5 + \cdots$$

Case of 6a

$$\frac{1}{t_{6a}} \left( 1 - \frac{12\sqrt{-3}}{t_{6a}} \right)^{\frac{1}{2}} F_{6a}^{6} = q^{1/2} + 15q^{3/2} + 54q^{5/2} - 88q^{7/2} - 423q^{9/2} + \cdots$$

$$\left( 1 - \frac{6\sqrt{-3}}{t_{6a}} \right) \frac{1}{t_{6a}} \left( 1 - \frac{12\sqrt{-3}}{t_{6a}} \right)^{\frac{1}{3}} F_{6a}^{8} = q^{1/2} - 27q^{3/2} - 378q^{5/2} - 832q^{7/2} + \cdots$$

$$\frac{1}{t_{6a}} \left( 1 - \frac{12\sqrt{-3}}{t_{6a}} \right)^{\frac{7}{6}} F_{6a}^{10} = q^{1/2} + 39q^{3/2} + 630q^{5/2} + 5336q^{7/2} + \cdots$$

$$\frac{1}{t_{6a}^{3}} \left( 1 - \frac{12\sqrt{-3}}{t_{6a}} \right)^{\frac{1}{6}} F_{6a}^{10} = q^{3/2} - 3q^{5/2} - 27q^{7/2} + 147q^{9/2} + \cdots$$

$$\frac{1}{t_{6a}} \left( 1 - \frac{6\sqrt{-3}}{t_{6a}} \right) \left( 1 - \frac{12\sqrt{-3}}{t_{6a}} \right) F_{6a}^{12} = q^{1/2} - 3q^{3/2} - 810q^{5/2} - 14848q^{7/2} + \cdots$$

$$\frac{1}{t_{6a}^{3}} \left( 1 - \frac{6\sqrt{-3}}{t_{6a}} \right) F_{6a}^{12} = q^{3/2} - 45q^{5/2} + 297q^{7/2} - 759q^{9/2} + \cdots$$

$$\frac{1}{t_{6a}} \left( 1 - \frac{12\sqrt{-3}}{t_{6a}} \right)^{\frac{11}{6}} F_{6a}^{14} = q^{1/2} + 63q^{3/2} + 1782q^{5/2} + 29768q^{7/2} + \cdots$$

$$\frac{1}{t_{6a}^3} \left( 1 - \frac{12\sqrt{-3}}{t_{6a}} \right)^{\frac{5}{6}} F_{6a}^{14} = q^{3/2} + 21q^{5/2} + 117q^{7/2} - 261q^{9/2} + \cdots$$

$$\frac{1}{t_{6a}} \left( 1 - \frac{6\sqrt{-3}}{t_{6a}} \right) \left( 1 - \frac{12\sqrt{-3}}{t_{6a}} \right)^{\frac{5}{3}} F_{6a}^{16} = q^{1/2} + 21q^{3/2} - 666q^{5/2} - 34048q^{7/2} + \cdots$$

$$\frac{1}{t_{6a}^3} \left( 1 - \frac{6\sqrt{-3}}{t_{6a}} \right) \left( 1 - \frac{12\sqrt{-3}}{t_{6a}} \right)^{\frac{2}{3}} F_{6a}^{16} = q^{3/2} - 21q^{5/2} - 567q^{7/2} - 2463q^{9/2} + \cdots$$

$$\frac{1}{t_{6a}^3} \left( 1 - \frac{12\sqrt{-3}}{t_{6a}} \right)^{\frac{3}{2}} F_{6a}^{18} = q^{3/2} + 45q^{5/2} + 837q^{7/2} + 7971q^{9/2} + \cdots$$

$$\frac{1}{t_{6a}^2} \left( 1 - \frac{6\sqrt{-3}}{t_{6a}} \right) \left( 1 - \frac{12\sqrt{-3}}{t_{6a}} \right)^{\frac{4}{3}} F_{6a}^{20} = q^{3/2} + 3q^{5/2} - 855q^{7/2} - 19719q^{9/2} + \cdots$$

#### References

- [1] M. Brouè and M. Enguehard, Polynômes des poids de certain codes et fonctions theta de certain reseaux, Ann. Sci. Ecole Norm. Sup. 6 (1973), 157–181.
- [2] J.M. Borwein and P.B. Borwein, A cubic counterpart of Jacobi's identity and the AGM, Trans. A. M. S. 323 (1991), 691–701.
- [3] J.M. Borwein and P.B. Borwein, Pi and the AGM- A study in analytic number theory and computational complexity, Wiley, 1987.
- [4] J. Conway and S. Norton, Monstrous Moonshine, Bull.London Math. Soc. 11 (1979) ,308-339.
- [5] J. Conway and N.J.A. Sloane, Sphere packings, Lattices and Groups, Springer-Verlag, 1988.
- [6] W. Ebeling, Lattices and Codes, Vieweg, 1994.
- [7] A. Erdélyi et al., Higher Transcendental Functions, vol. 1, McGraw-Hill, New York, 1953.

- [8] D. Ford, J. Mckay and S. Norton, More on replicable functions Comm. Alg. 22 (1994), 5175-5193.
- [9] J. Harnad and J. Mckay, Modular solutions to equations of generalized Halphen type, preprint.

  ArXiv 1998; published 2000
- [10] Y. Ihara, Schwarzian equations, J. Fac. Sci. Univ. Tokyo, Sec. 1A 21-1 (1974), 97-118.
- [11] M. Kaneko and D. Zagier, Supersingular j-invariants, hypergeometric series and Atkin's orthogonal polynomials, AMS / IP Studies in Advanced Mathematics, 7 (1998), 97–126.
- [12] M. Koike, A note on hypergeometric polynomoials over the finite field, Proc.Jangjunn Intern. Conf. of Math. Sciences, 1996.
- [13] B. Lian and S.-T. Yau, Arithmetic Properties of Mirror Maps and Quantum Couplings, hep-th/9506112.
- [14] F.J. MacWilliams, C.J. Mallows and N.J. Sloane, Generalizations of Gleason's theorem on weight enumerators of self-dual codes, IEEE Trans. Information Theory, 18 (1972), 794–805.
- [15] D.P. Maher, Modular forms from codes, Canada. J. Math. 32 (1980), 40-58.
- [16] N. Sloane, Self-dual codes and lattices, Proc. Symp. in Pure Math. A. M. S. 34 (1979), 273–308.
- [17] N. Sloane, Codes over GF(4) and complex lattices, J. Alg. 52 (1978), 168-181.
- [18] P. Stiller, Classical Automorphic Forms and Hypergeometric Functions, J. Number Theory 28 (1988), 219–232.
- [19] K. Takeuchi, A characterization of arithmetic Fuchsian groups, J. Math. Soc. Japan, 27 (1975), 600-612.

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