Gauss's continued fraction for the function \( \text{arctanh}(z) \) is
\[
\frac{z}{1 - \frac{1^2 z^2}{(3 - \frac{2^2 z^2}{(5 - \frac{3^2 z^2}{(7 - \ldots ))})})} \ldots \quad (1)
\]
valid for complex \( z \) not in either of the intervals \((-\infty, -1]\) or \([1, \infty)\).

In this note we find expressions in terms of Legendre polynomials for both the numerator and denominator polynomials of the \( n \)-th convergent of Gauss's continued fraction. This allows us to give rapidly converging series for some well-known constants.

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We begin by replacing \( z \) with \( 1/z \) in (1) and then making use of equivalence transformations to obtain the continued fraction representation
\[
\text{arctanh}(1/z) = \frac{1}{z - \frac{1}{3/z - \frac{2^2}{5/z - \frac{3^2}{7/z - \ldots - \frac{(n - 1)^2}{((2n - 1)/z - \ldots )})}}}} \ldots \quad (2)
\]
valid for complex \( z \) not in the closed interval \([-1, 1]\).

Let \( N(n, z)/D(n, z) \) denote the \( n \)-th convergent to the continued fraction (2):
\[
N(n, z)/D(n, z) = \frac{1}{z - \frac{1}{3/z - \frac{2^2}{5/z - \frac{3^2}{7/z - \ldots - \frac{(n - 1)^2}{((2n - 1)/z - \ldots )})}}}}.
\]
The first four convergents (numbered 1 through 4) are
\[
1/z, \quad 3/z/(3/z^2 - 1), \quad z*(4/z^2 - 15)/(3*(3/z^2 - 5)) \quad \text{and}
\]
By the elementary theory of continued fractions, both the sequence of numerator polynomials \{N(n, z)\} and the sequence of denominator polynomials \{D(n, z)\} satisfy the 3-term recurrence
\[
u(n, z) = (2n - 1)z\nu(n-1, z) - (n - 1)^2\nu(n-2, z) \quad \ldots \quad (3)
\]
for \(n \geq 3\), with the initial values
\[N(1, z) = 1, \quad N(2, z) = 3z\]
and
\[D(1, z) = z, \quad D(2, z) = 3z^2 - 1.\]

The following theorem gives explicit expressions for the polynomials \(N(n, z)\) and \(D(n, z)\) in terms of Legendre polynomials.

**Theorem.** Let \(P(n,z)\) denote the \(n\)-th Legendre polynomial. Then

(i) \(D(n, z) = n!\times P(n, z)\)

(ii) \(N(n, z) = D(n, z) \times \sum_{k = 1..n} 1/(k\times P(k-1, z)\times P(k, z))\)

**Proof.**

The Legendre polynomials satisfy the 3-term recurrence
\[
n \times P(n, z) = (2n - 1)z\times P(n-1, z) - (n - 1)\times P(n-2, z) \quad \ldots \quad (4)
\]
with \(P(1, z) = z\) and \(P(2, z) = (3z^2 - 1)/2\). Thus (i) holds for \(n = 1\) and \(n = 2\).

Multiplying (4) by \((n - 1)!\) we see that the polynomial sequence \(\{n!\times P(n, z)\}\) satisfies the same recurrence (3)
\[
u(n, z) = (2n - 1)z\nu(n-1, z) - (n - 1)^2\nu(n-2, z)
\]
satisfied by the denominator polynomials \(D(n, z)\), and with the same initial conditions.

Thus the polynomial sequences \(\{D(n, z)\}\) and \(\{n!\times P(n, z)\}\) are identical, completing the proof of (i).
(ii) Define

\[ A(n, z) = D(n, z) \times \sum_{k = 1..n} \frac{1}{(k \times P(k-1, z) \times P(k, z))} \]  \hspace{1cm} \text{(5)}

We calculate the initial values

\[ A(1, z) = 1 = N(1, z) \]

and

\[ A(2, z) = 3z = N(2, z). \]

We show that the sequence \( \{A(n, z)\} \) also satisfies the
3-term recurrence (3) satisfied by the sequence \( \{N(n, z)\} \), hence
proving that \( A(n, z) = N(n, z) \) for all \( n \).

From (5),

\[ A(n+1, z) = D(n+1, z) \times \sum_{k = 1..n+1} \frac{1}{(k \times P(k-1, z) \times P(k, z))} \]

\[ = D(n+1, z) \times \sum_{k = 1..n} \frac{1}{(k \times P(k-1, z) \times P(k, z))} \]

\[ + \frac{D(n+1, z)}{(n + 1) \times P(n, z) \times P(n+1, z)} \]

\[ = \left( \frac{D(n+1, z)}{D(n, z)} \right) \times A(n, z) \]

\[ + \frac{D(n+1, z)}{(n + 1) \times P(n, z) \times P(n+1, z)}. \]

Substituting the value \( D(n, z) = n! \times P(n, z) \) from part (i) and
multiplying both sides of the resulting identity by \( P(n, z) \) we
find that

\[ P(n, z) \times A(n+1, z) = (n + 1) \times P(n+1, z) \times A(n, z) + n!. \]  \hspace{1cm} \text{(6)}

Hence

\[ P(n+1, z) \times A(n+2, z) = (n + 2) \times P(n+2, z) \times A(n+1, z) + (n + 1)! \]  \hspace{1cm} \text{(7)}

Multiply (6) by \( n + 1 \), subtract the result from (7) and then
replace \( n \) with \( n - 2 \). Making use of the recurrence equation
(4) for the Legendre polynomials we find after a short calculation
that \( A(n, z) \) satisfies the same 3-term recurrence (3)

\[ A(n, z) = (2n - 1)z \times A(n-1, z) - (n - 1)^2 \times A(n-2, z) \]
satisfied by the numerator polynomials \(N(n, z)\), completing the proof of part (ii).

**Corollary 1.**

\[
\text{arctanh}(1/z) = \lim_{n \to \infty} \frac{N(n, z)}{D(n, z)} = \sum_{k \geq 1} \frac{1}{k \cdot P(k, z) \cdot P(k-1, z)}
\]

valid for complex \(z\) not in the closed interval \([-1, 1]\).

This result allows us to give rapidly converging series for values of some well-known constants, for example,

\[
i \cdot \text{artanh}(1/i) = \frac{\pi}{4} = \sum_{n \geq 1} \frac{i}{n \cdot P(n, i) \cdot P(n-1, i)},
\]

\[
2 \cdot \text{arctanh}(1/2) = \log(3) = 2 \cdot \sum_{n \geq 1} \frac{1}{n \cdot P(n, 2) \cdot P(n-1, 2)}
\]

and

\[
2 \cdot \text{arctanh}(1/3) = \log(2) = 2 \cdot \sum_{n \geq 1} \frac{1}{n \cdot P(n, 3) \cdot P(n-1, 3)}.
\]

The last result is due to Burnside.

**Corollary 2.**

The \(n\)-th convergent of Gauss’s continued fraction (1)

\[
z/(1 - 1^2z^2/(3 - 2^2z^2/(5 - \ldots (n - 1)^2z^2/(2n-1))))
\]

is equal to \(N(n, 1/z)/D(n, 1/z)\).

The finite continued fraction has a Taylor expansion around \(z = 0\) equal to

\[
z + z^3/3 + z^5/5 + z^7/7 + \ldots + z^{(2n-1)}/(2n - 1) + O(z^{(2n+1)}).
\]

Thus the rational function \(N(n, 1/z)/D(n, 1/z)\) is a Padé approximant to \(\text{arctanh}(z)\): more precisely, \(N(2n+1, 1/z)/D(2n+1, 1/z)\) is the \([2n+1, 2n]\) Padé approximant to \(\text{arctanh}(z)\) and \(N(2n, 1/z)/D(2n, 1/z)\) is the \([2n-1, 2n]\) Padé approximant to \(\text{arctanh}(z)\).