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A SET OF EIGHT NUMBERS

it can be shown for large n that \( Y_{n-1} \) is approximately equal to \((2\pi pq)^{-1/2}, or 1/\sqrt{2\pi n}\). Thus from (6) we obtain the approximation

\[
MD_n \sim \sqrt{2npq/\pi} = \sqrt{2/\pi} \sigma_n = 0.79788\sigma_n.
\]

More exact computation, using the remainder terms in Stirling's formula, yields the better approximation

\[
\frac{\pi}{2} (MD_n)^2 = npq + (np - [np])(nq - [nq]) - (1 - pq)/6 + E_n/24n,
\]

where the error coefficient \( E_n \) becomes numerically less than or equal to unity as \( n \) becomes infinite, for all choices of \( np \) between 1 and \( n-1 \); and \([np]\) and \([nq]\) denote the greatest integers not exceeding \( np \) and \( nq \) respectively.

A SET OF EIGHT NUMBERS

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1. Introduction. In this paper the operation of adding the squared digits of any natural number \( A \) a finite number of times is proved to transform \( A \) either to unity or to one of a set of eight natural numbers closed under the operation.

2. Definitions. We use the expression natural number to denote a member of set 1, 2, 3, \( \cdots \) of positive integers. Zero has not been adjoined to this set and is not to be included in the definition.

The operator \( G \) is defined by the equation

\[
G(A) = \sum_{i=1}^{R} X_i^2,
\]

where \( A \) is a natural number of \( R \) digits given by

\[
A = \sum_{i=1}^{R} X_i10^{i-1}.
\]

Since \( A \) has \( R \) digits, \( X_R \neq 0 \).

We note that \( G(0) = 0 \), and \( G(1) = 1 \).

Using the customary notation, we write \( G^n(A) \), where \( n > 1 \), for \( n \) successive applications of the operator \( G \) to \( A \).

\( G \) is not a linear operator since, in general, \( G(A_1 + A_2) \neq G(A_1) + G(A_2) \).

The set of numbers

\[
\begin{align*}
    a_1 &= 4, & a_5 &= 89, \\
    a_2 &= 16, & a_6 &= 145, \\
    a_3 &= 37, & a_7 &= 42, \\
    a_4 &= 58, & a_8 &= 20,
\end{align*}
\]
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is closed under the operation defined by (1). We call (3) *Set K*, and use the symbol $a'$ to denote any non-specified element of the set. The equation

$$G^r(a') = a'$$

is easily verified.

Numbers of the form $10^n$, $10 \cdot 10^n$, $10^{n+1} + 3$, where $n$ is a positive integer or zero, and others not specified here, satisfy the equation

$$G^r(A) = 1$$

for some integer $r > 0$. Any natural number satisfying (5) will be denoted by the symbol $b'$.

3. Preliminary Lemmas. In what follows, the symbols $A$ and $B$ always represent finite natural numbers in the denary system of notation.

**Lemma 1.** Any natural number $A$ of $R$ digits, where $R \geq 4$, satisfies the inequality

$$G(A) < A.$$  

It is evident that $G(A) \leq 81R$, and that $A \geq 10^{R-1}$. The inequality

$$81R < 10^{R-1}$$

becomes, upon taking the common logarithm of each member and transposing,

$$\log_{10} R < R - 2.9085,$$

an inequality valid for $R \geq 4$.

**Lemma 2.** For any natural number $A$ there exists a positive integer $n$ such that

$$G^n(A) \leq 162.$$  

For $R \geq 4$, Lemma 1 establishes the inequality (6). As a direct consequence of (6), the operator $G$ applied to $A$ a finite number of times must result in a natural number of less than four digits, since for $R = 4, G(A) \leq 324$.

For $R < 4$, the following inequalities are readily established.

$$G(A) \leq 243,$$

$$G^2(A) \leq G(199) = 163,$$

$$G^3(A) \leq G(99) = 162.$$  

Since $G(A)$, where $A$ is a three digit number, cannot exceed $3 \cdot 81 = 243$, (10) is obviously valid. Also, since $G(199) \geq G(B)$ for any $B \leq 243$, (11) holds. Finally, since $G(99) \geq G(P)$ for any $P \leq 163$, (12) is proved.

The inequalities (10), (11), and (12) complete the proof of Lemma 2.

4. Convergence of $G^n(A)$. The following theorem is the main result of this paper.
Theorem 1. For every natural number \( A \) there exists either a positive integer \( n \) such that (5) holds for all \( r \geq n \), or a positive integer \( m \) such that

\[
G^r(A) = a'
\]

for all \( r \geq m \), where \( a' \) is some element of Set \( K \).

From Lemma 2 it is evident we need prove the theorem only for \( A \leq 162 \). The writer was unable to find a simple indirect proof sufficiently superior to the following direct one of selective verification to justify its inclusion here.

We consider two cases.

Case 1. \( 100 \leq A \leq 162 \).

For \( A \) thus restricted, it is apparent that \( G(A) \leq G(159) = 107 \). Direct application of the operator \( G \) to \( A \) over the range 100 to 107 gives

\[
\begin{align*}
G(100) &= 1, & G^2(104) &= a' = 89, \\
G^2(101) &= a' = 4, & G^2(105) &= a' = 16, \\
G(102) &= a' = 89, & G(106) &= a' = 37, \\
G^2(103) &= 1, & G^6(107) &= a' = 89,
\end{align*}
\]

thus completing the proof of the theorem for Case 1.

Case 2. \( 0 < A < 100 \).

For \( A = 10X + Y \), where \( 0 \leq X \leq 9 \), and \( 0 \leq Y \leq 9 \), the following identity is valid.

\[
G(10X + Y) = G(10Y + X).
\]

Further, if \( G^n(A) = a' \), and \( G^m(B) = A \), it follows that there exists a number \( h = n + m \) such that \( G^h(B) = a' \).

By means of these considerations, it is possible to verify Theorem 1 numerically for all \( A < 100 \) by actual computation of \( G^n(A) \) for 30 values of \( A < 100 \), thus completing the proof of the theorem.

The writer is aware of the inelegance of such a proof, and would like very much to see a simple indirect one. However, proving the non-existence of another set like (3), which seems a necessary step, is quite difficult because of the non-linear character of \( G \).

Corollary. For every natural number \( A \) there exists either a positive integer \( n \) such that \( G^n(A) = 1 \), or a positive integer \( m \) such that \( G^m(A) = 4 \).

The corollary follows directly from Theorem 1 and the nature of Set \( K \). Since every natural number is transformed either into unity or into an element of Set \( K \) by the operator \( G \), we need only note that for every \( a' \neq 4 \), there exists a positive integer \( r \leq 7 \) such that \( G^r(a') = 4 \).

Theorem 2. The number of digits \( N \) in \( G(A) \), where \( A \) has \( R \) digits, satisfies the inequality

\[
N \leq 2.9 + \log_{10} R.
\]
This theorem is a simple consequence of the inequality \( G(A) \leq 81R \). We have
\[
G(A) \leq 10^{1.9} + \log_{10} R
\]
a number of \( N \) digits, where \( N \leq 2.9 + \log_{10} R \).

**Theorem 3.** The only solutions in natural numbers of
\[
G^n(A) = A,
\]
where \( n \geq 1 \), are
\[
(19) \quad A = 1, \quad n = J,
\]
\[
(20) \quad A = a', \quad n = 8,
\]
where \( J \) is any natural number.

If we assume the existence of a natural number \( A > 1 \) and different from \( a' \) such that \( G^n(A) = A \) for some \( n \geq 1 \), it follows that \( A \) would not be transformed into either unity or an element of Set \( K \) by a finite number of applications of the operator \( G \) to \( A \). But this is a direct contradiction of Theorem 1, and hence the assumption is false.

**5. Concluding Remarks.** A problem suggested by the one just discussed is that of repeatedly summing the cubed digits of a natural number. A complication occurs, however, since there is more than one number \( A \) such that \( H(A) = A \), where \( H \) is the operator analogous to (1) given by
\[
H(A) = \sum_{i=1}^{R} X_i^3.
\]
For example, \( H(153) = 153 \), \( H(407) = 407 \), and \( H(371) = 371 \). This destroys the factor of uniqueness, since \( H(A) \) may be unity as when \( A = 100 \); or \( A \) may be transformed into a number \( A' \) like 153.

It is interesting to note that since for any number \( A \) transformed into some element of Set \( K \) by a finite number of applications of \( G \) we can construct a number \( B = 10^4 \) such that \( G(B) = 1 \), there are at least "as many" numbers satisfying (5) as (13). This intuitually unsatisfying conclusion results from the comparison of two infinite sets.

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Leibniz discovers the obvious. I have made some observations on prime numbers which, in my opinion, are of consequence for the perfection of the science of numbers . . . . If the sequence [of primes] were well known, it would enable us to uncover the mystery of numbers in general; but up till now it has seemed so bizarre that nobody has succeeded in finding any affirmative characteristic or property . . . . I believe I have found the right road for penetrating their [primes'] nature: but not having had the leisure to pursue it, I shall give you here a positive property, which seems to me curious and useful.—Leibniz, in a letter to the editor of the *Journal des Savants*, 1678.

The discovery: a prime is necessarily of one or other of the forms \( 6n + 1, \ 6n + 5 \).—Contributed.