Formula for A003324

OEIS A003324: Let $b(0)$ be the sequence 1,2,3,4. Proceeding by induction, let $b(n)$ be a sequence of length 2^{n+2}. Quarter $b(n)$ into four blocks A, B, C, D each of length 2^n, so that $b(n) = ABCD$, then $b(n+1) = ABCDADCB$.

Write $a(n)$ as the n-th term of A003324.

Theorem. $a(n) = n \bmod 4$ for odd n; for even n, write $n = (2k+1) \times 2^e$, then $a(n) = 2$ if $k+e$ is odd, $a(n) = 4$ if $k+e$ is even.

Proof. Define $s(n) = n \bmod 4$ for odd n, $s(n) = 2$ for $n = (2k+1) \times 2^e$ with odd $k+e$, $s(n) = 4$ for $n = (2k+1) \times 2^e$ with even $k+e$, then our goal is to show $a(n) = s(n)$ for all n. We shall prove this by induction.

For $n = 1, 2, 3, 4$, we have $a(n) = n = s(n)$. Suppose $a(n) = s(n)$ for $n \leq 2^N$, $N \geq 2$. By definition we have

$$a(2^N + m) = \begin{cases} a(m), & 0 < m \leq 2^{N-2} \text{ or } 2^{N-1} < m \leq 3 \times 2^{N-2}, \\ a(m + 2^{N-1}), & 2^{N-2} < m \leq 2^{N-1}, \\ a(m - 2^{N-1}), & 3 \times 2^{N-2} < m \leq 2^N. \end{cases}$$

For convenience, define

$$\varphi_N(m) = \begin{cases} m, & 0 < m \leq 2^{N-2} \text{ or } 2^{N-1} < m \leq 3 \times 2^{N-2}, \\ m + 2^{N-1}, & 2^{N-2} < m \leq 2^{N-1}, \\ m - 2^{N-1}, & 3 \times 2^{N-2} < m \leq 2^N. \end{cases}$$

Then we have $a(2^N + m) = a(\varphi_N(m)) = s(\varphi_N(m))$ for $m \leq 2^N$. So we just have to show $s(2^N + m) = s(\varphi_N(m))$ for $m \leq 2^N$.

The case where $2^N + m$ is odd is easy: for odd $2^N + m$, it suffices to show $2^N + m \equiv \varphi_N(m) \pmod 4$. If $N \geq 3$, this is obviously true. If $N = 2$, then $\varphi_N(m) = m \equiv 2^N + m \pmod 4$.

For even $2^N + m$, write $m = (2k+1) \times 2^e \leq 2^N$. If $e \leq N-2$, then $s(2^N + m) = s\big((2(k + 2^{N-e-1}) + 1) \times 2^e\big)$ and $s(\varphi_N(m)) = s(m + \varepsilon 2^{N-1}) = s\big((2(k + \varepsilon 2^{N-e-2}) + 1) \times 2^e\big)$ for some $\varepsilon \in \{-1, 0, 1\}$. Since $k + 2^{N-e-1} + e \equiv k + \varepsilon 2^{N-e-2} + e \pmod 2$ (if $e = N-2$, then $m = 2^{N-2}$ or $3 \times 2^{N-2}$, so $\varepsilon = 0$), we have $s(2^N + m) = s(\varphi_N(m))$.

If $e \geq N-1$, the only possibilities are $m = 2^{N-1}$ or $m = 2^N$.

- If $m = 2^{N-1}$, then $2^N + m = (2 \times 1 + 1) \times 2^{N-1}$, $\varphi_N(m) = (2 \times 0 + 1) \times 2^N$, since $1 + (N-1) = 0 + N$, we have $s(2^N + m) = s(\varphi_N(m))$.
- If $m = 2^N$, then $2^N + m = (2 \times 0 + 1) \times 2^{N+1}$, $\varphi_N(m) = (2 \times 0 + 1) \times 2^{N-1}$, since $0 + (N+1) \equiv 0 + (N-1) \pmod 2$, we also have $s(2^N + m) = s(\varphi_N(m))$.

So $s(2^N + m) = s(\varphi_N(m))$ for $m \leq 2^N$. Then we have $a(n) = s(n)$ for $n \leq 2^{N+1}$, by induction, the formula is proved. ∎