Formula for A003324

OEIS A003324: Let \( b(0) \) be the sequence 1,2,3,4. Proceeding by induction, let \( b(n) \) be a sequence of length \( 2^{n+2} \). Quarter \( b(n) \) into four blocks \( A, B, C, D \) each of length \( 2^n \), so that \( b(n) = ABCD \), then \( b(n + 1) = ABCDADCB \).

Write \( a(n) \) as the \( n \)-th term of A003324.

**Theorem.** \( a(n) = n \mod 4 \) for odd \( n \); for even \( n \), write \( n = (2k + 1) \times 2^e \), then \( a(n) = 2 \) if \( k + e \) is odd, \( a(n) = 4 \) if \( k + e \) is even.

**Proof.** Define \( s(n) = n \mod 4 \) for odd \( n \), \( s(n) = 2 \) for \( n = (2k + 1) \times 2^e \) with odd \( k + e \), \( s(n) = 4 \) for \( n = (2k + 1) \times 2^e \) with even \( k + e \), then our goal is to show \( a(n) = s(n) \) for all \( n \). We shall prove this by induction.

For \( n = 1, 2, 3, 4 \), we have \( a(n) = n = s(n) \). Suppose \( a(n) = s(n) \) for \( n \leq 2^N \), \( N \geq 2 \). By definition we have

\[
a(2^N + m) = \begin{cases} 
a(m), & 0 < m \leq 2^{N-2} \text{ or } 2^{N-1} < m \leq 3 \times 2^{N-2}, \\
a(m + 2^{N-1}), & 2^{N-2} < m \leq 2^{N-1}, \\
a(m - 2^{N-1}), & 3 \times 2^{N-2} < m \leq 2^N. 
\end{cases}
\]

For convenience, define

\[
\varphi_N(m) = \begin{cases} 
m, & 0 < m \leq 2^{N-2} \text{ or } 2^{N-1} < m \leq 3 \times 2^{N-2}, \\
m + 2^{N-1}, & 2^{N-2} < m \leq 2^{N-1}, \\
m - 2^{N-1}, & 3 \times 2^{N-2} < m \leq 2^N. 
\end{cases}
\]

Then we have \( a(2^N + m) = a(\varphi_N(m)) = s(\varphi_N(m)) \) for \( m \leq 2^N \). So we just have to show \( s(2^N + m) = s(\varphi_N(m)) \) for \( m \leq 2^N \).

The case where \( 2^N + m \) is odd is easy: for odd \( 2^N + m \), it suffices to show \( 2^N + m \equiv \varphi_N(m) \) (mod 4). If \( N \geq 3 \), this is obviously true. If \( N = 2 \), then \( \varphi_N(m) = m \equiv 2^N + m \) (mod 4).

For even \( 2^N + m \), write \( m = (2k + 1) \times 2^e \leq 2^N \). If \( e \leq N - 2 \), then \( s(2^N + m) = s((2k + 2^{N-e-1} + 1) \times 2^e) \) and \( s(\varphi_N(m)) = s(m + 2^{N-e}) = s((2(k + 2^{N-e-2}) + 1) \times 2^e) \) for some \( e \in \{-1, 0, 1\} \). Since \( k + 2^{N-e-1} + 1 \equiv k + 2^{N-e-2} + e \) (mod 2) (if \( e = N - 2 \), then \( m = 2^{N-2} \) or \( 3 \times 2^{N-2} \), so \( e = 0 \)), we have \( s(2^N + m) = s(\varphi_N(m)) \).

If \( e \geq N - 1 \), the only possibilities are \( m = 2^N - 1 \) or \( m = 2^N \).

- If \( m = 2^N - 1 \), then \( 2^N + m = (2 \times 1 + 1) \times 2^N - 1 \), \( \varphi_N(m) = (2 \times 0 + 1) \times 2^N \), since \( 1 + (N - 1) = 0 + N \), we have \( s(2^N + m) = s(\varphi_N(m)) \).

- If \( m = 2^N \), then \( 2^N + m = (2 \times 0 + 1) \times 2^{N+1} \), \( \varphi_N(m) = (2 \times 0 + 1) \times 2^{N-1} \), since \( 0 + (N + 1) \equiv 0 + (N - 1) \) (mod 2), we also have \( s(2^N + m) = s(\varphi_N(m)) \).

So \( s(2^N + m) = s(\varphi_N(m)) \) for \( m \leq 2^N \). Then we have \( a(n) = s(n) \) for \( n \leq 2^{N+1} \), by induction, the formula is proved. \( \square \)