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D E Knuth
H. Wilf
C. L. Mallows
D. Williams

correspondence
1974

8 pages total
Dr. Neil J. A. Sloane  
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Dear Neil,

I probably forgot to tell you about the sequence 1, 1, 2, 4, 14, 62, … that I published in the Monthly long ago (April 1974, page 340). It counts “necklace permutations,” a fairly natural kind of combinatorial object that I hope somebody will soon enumerate.

Cordially,

Donald E. Knuth  
Professor

DEK/pw
February 26, 1973

Prof. David A. Klarner
Computer Science Dept.
Stanford University

Dear David:

Last week while skiing in Norway I thought of another enumeration problem I couldn’t solve, but it looks interesting so I wonder if you will see how to do it.

The problem is to discover the number of different "necklace permutations" -- this is a word I made up, you can change it if you wish. It represents the number of essentially different orders in which a person can change n white beads of a necklace into all black beads, not counting the operation of turning and/or flipping over the necklace whenever such an operation preserves the current black/white pattern.

Thus, \( a_1 \ldots a_n \) is equivalent to \((\varepsilon a_{k+1}) \ldots (\varepsilon a_{k+j})a_{k+1} \ldots a_n \mod n \), whenever both are permutations, for all \( \varepsilon = 1 \), and \( 1 \leq j, k \leq n \); and transitivity applies too. In particular, \( a_1 \ldots a_n \) is always equivalent to \( a_2 a_1 a_3 \ldots a_n \) and to \( a_1 \ldots a_{n-1} a_n \). The distinct necklace permutations for \( n \leq 6 \) are

\[
\begin{align*}
1; \\
12; \\
123; \\
1234, 1324; \\
12345, 12354, 12435, 13245, 13425; \\
123456, 123546, 124356, 124536, 124635, 132456, 132546, \\
132456, 134256, 134625, 135246, 142356, 142536, 142653; \\
\end{align*}
\]

and if my quick count isn’t wrong there are 62 of order 7.

Sincerely,

Donald E. Knuth
Professor

\[
1, 1, 1, 2, 4, 14, 62;
\]
Dear Dan:

Thanks for unearthing the letters to Kleitman and Klarner, from 1972-3. I was aware of the fact that the # of terms (\(f(n)\)) that certain patterns are "near" but not quite, independent of the pattern. I made tables about 5 years ago, and conjectured that the pattern of length \(k\) is fixed, and \(f(n)\) is the # of patterns of length \(k\) that contain the pattern, then the asymptotic expansion of \(f(n)\) will have a first term that is independent of the other patterns. This is still unsettled. I've thought about it a bit, asked about it. The asymptotics is known of the pattern - identity, others about it. It's all tied to stack-sortable permutations.

Your note to Klarner has just disrupted my lunch hour.

First let me transform the question.

For \(n\), let \(T_n\) be the set of all necklaces of \(n\) beads of 2 colors (permutation classes under, say, de Bruijn group, as you prescribed).

Then \(T_n\) into a partially ordered set, as follows: a necklace \(v\) is covered by a necklace \(w\) if \(v\) can be transformed into \(w\) by a single head-blackening. The figure of p.1, included, shows the single head-blackening.

Then your question is:

\(\text{ how many paths are there, in } T_n, \text{ from } \phi? \) (colour \(n=6\)).

This kind of question has led to many deep combinatorial results. The weak Bruhat order of all permutations, where \(w' \leq w\) means that \(w'\) can be obtained from \(w\) by a finite sequence of adjacent transpositions, each of which increases the # of inversions, is \(\overline{\text{f(n)} = \#\text{ of paths from } (2-n) \to (n-1 \cdots 1)}\).

Then Stanley "observed" that \(f(n) = \#\text{ of } V.T.\text{ on the shade}\).

They proved it algebraically. Edelman & Greene proved it algebraically. It turns out that the whole thing is implicitly contained in some older work of Schiffler on quadratic monoids.
To get back to this question, I checked Sloane to see if 1, 1, 1, 2, 4, 14, 62,... is the beginning of some famous sequence, and it isn't. Nevertheless, on page 1 enclosed I sketched the matrix of its (above diagonal) entries, which has $(n^2)$ entries equal to the number of paths in question, as illustrated on page 2, enclosed.

Naturally, I haven't answered anything. But somehow I feel that I know what the question "really" is. It suggests that instead of diving in feet first to the deeper waters of the n-dimensional cube, why not stick with the cubic groups, move, and stick with cyclic equivalence. That one is plenty hard, still, but might be doable. The sequence begins 1, 1, 2, 4, 23, ... (R. Sloane).

Curtis Greene has done not only the number of maximal chains from bottom to top in the poset B_n, but also any more general theorems in the "shuffle" proof. So maybe I'll ask him if he knows any more general theorems.

Anyway, that's what happened during my lunch break today.

Best,

[Signature]

[Handwritten] (Wilf)
the 14 paths from bottom to top (path = sequence of edge numbers)

n = 6
\[ A \text{ matrix, } \sqrt{n} \ n = 6 \]

\[ (A^6) = 14 \]
November 8, 1990

Here is another installment of ruminations on the partially ordered set of necklaces of \( n \) beads (read this after the batch of handwritten stuff I sent you yesterday).

When I looked at that poset, it occurred to me was that I didn't even know the numbers of necklaces in its layers, i.e., the numbers of necklaces of \( n \) beads, each black or white, with exactly \( k \) black beads. This is quite a fundamental number, because it's the number of subsets of \( k \) unlabelled things chosen from a set of \( n \) unlabelled things on a circle. So it is a circular binomial coefficient, which is really quite nice.

I'll denote it by \( \binom{n}{k}{_k} \), where the subscript indicates that the operative group is the cyclic group. The layer counts in your original question would be \( \binom{n}{k}{_D} \), for the dihedral group. The original binomial coefficients belong to the identity group \( E_n \), etc. etc. (I think it's delightful that all of a sudden there are mountains of new binomial coefficients to play with).

In general, if \( G \) is any subgroup of \( S_n \) then by Pólya's theorem one has the generating function (I'm following Harary Graphical Enumeration, p. 36)

\[
\sum_k \binom{n}{k}{_G} z^k = Z(G, 1 + z) \tag{1}
\]

where \( Z \) is the cycle index of \( G \). Precisely, \( Z = Z(s_1, \ldots, s_n) \) is a polynomial in \( n \) variables, and we are to substitute \( 1 + z^d \) for each \( s_j \) on the right side of (1).

In particular, since

\[
Z(C_n, s) = \frac{1}{n} \sum_{d|n} \phi(d) s_d^{n/d}
\]

one has

\[
\sum_k \binom{n}{k}{_C} z^k = \frac{1}{n} \sum_{d|n} \phi(d)(1 + z^d)^{n/d}
\]

and therefore the evaluation

\[
\binom{n}{k}{_C} = \frac{1}{n} \sum_{d|n, (d,k)} \phi(d) \binom{n/d}{k/d}
\]

The beginnings of the cyclic Pascal triangle are as follows:

\[
\begin{array}{ccccccccc}
1 \\
1 \\
1 1 \\
1 1 1 \\
1 1 1 1 \\
1 1 2 1 1 \\
1 1 2 2 1 1 \\
1 1 3 4 3 1 1 \\
1 1 3 5 5 3 1 1 \\
1 1 4 7 10 7 4 1 1 \\
1 1 4 10 14 14 10 4 1 1 \\
1 1 5 12 22 26 22 12 5 1 1 \\
\end{array}
\]

Dennis White has a theorem, that I'll look up, to the effect that Pólya theory tends to make unimodal sequences. It might imply already that the rows are always unimodal.

This is really pretty stuff, and I'm glad you sent me your original question.

(i.e. # of distinct ways of successively coloring all the beads of a necklace, ignoring rotation & flipping.)

\[ n = 4 \]

\[
\begin{array}{c}
\text{# = 2} \\
\text{(Mallows)}
\end{array}
\]

\[ n = 3 \]

\[
\begin{array}{c}
\text{# = 1} \\
\text{(Mallows)}
\end{array}
\]