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John U. Marshall

The Construction of the Löschian Landscape

Previous descriptions of Lösch's economic landscape have not included accurate rules for maximizing the spatial coincidence of suppliers of different goods. With N *denoting the number of farms served by a supplier, the correct procedure is shown to involve separate treatment of (a) cases in which* N *is of the form* x^2 *or* $3x^2$, *(b) cases in which* N *is a prime number, and (c) all other cases. In the first case the primitive function points of the lattice of suppliers lie on the boundaries of the landscape's conventional subsectors. In the second case the locations for the primitive function points may be chosen at will, but in the third case these locations are completely determined by the locations selected for cases in which* N *is prime. For all lattices with two or more possible orientations the grid coordinates of the primitive function points correspond to unequivalent solutions of the Diophantine equation used to generate the Löschian numbers. It is also shown that city-rich and city-poor sectors do not exist in the Löschian landscape whether or not the spatial coincidence of suppliers is maximized.*

When Lösch developed the theoretical economic landscape that bears his name, he considered it to be relevant for all nonextractive punctiform production, including manufacturing as well as central-place activities [*11*, pp. 101-37]. It has long been recognized, however, that the Löschian landscape is a tenable construct only for those types of production in which no significant locational attraction is exerted by raw materials [*8*, p. 274]. Accordingly, Lösch's approach is regarded as inadequate with respect to manufacturing, and the conventional view is to accept his model as a contribution to central-place theory [*1*, pp. 68-73; 7, pp. 122-24; *9*, pp. 96-98]. It almost goes without saying that central-place theory refers to only one component, albeit a very important component, of the total urban economy.

Central-place theory is normally written for an economy in which each entrepreneur provides only a single good or service. Real central-place firms, by contrast, rarely deal in just one good. There is no reason why classical central-place theory should not incorporate multiple-good

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firms, provided that each entrepreneur is assumed to view his total inventory as an integral package when deciding where to locate. For convenience of exposition, the traditional fiction of the single-good firm is retained in this paper, but it is stressed that the introduction of sets of identical multiple-good firms, one set for each central-place function, would not affect any of the conclusions.

The focus of this paper is the precise manner in which the Löschian landscape is constructed. Lösch's own description is less than complete, and later writers, in the words of Tarrant, "have rather avoided the problem by a bold statement of what is required" [*18*, p. 113]. The discussion that follows goes beyond the point reached in previous treatments and attempts to clarify several matters that have hitherto remained obscure, notably the supposed existence of city-rich and city-poor sectors in the finished landscape.

Let us assume a Löschian environment [*11*] in which individual farms are the Dirichlet regions of their respective farmsteads. Let the lattice of farmstead points be denoted by $L(1)$. The complete set of integers Q denoting the sizes of each market area N is generated by the cosine rule [*3, 4, 18*]. For example, let M and J in Figure 1 represent the locations of two adjacent suppliers of the same good. Measuring from M in terms of the distance between neighboring farmsteads, the point J has the coordinates (2, 5), whence $x = 2$, $y = 5$, and

$$N = x^2 + xy + y^2 = 39\,. \tag{1}$$

In other words, MJ is the distance separating adjacent suppliers of a good requiring 39 farms in its market area. It can be shown by elementary geometry that this distance is $\sqrt{39}$, or more generally that

$$D = d\sqrt{N}\,, \tag{2}$$

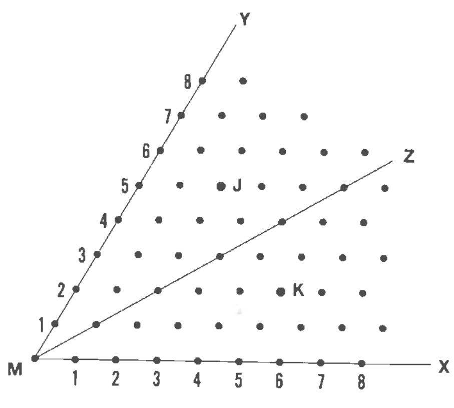

Fig. 1. Reference Grid for a Portion of the Löschian Landscape

where D is the distance between adjacent suppliers of a good requiring N farms in its market area, and where d, here taken as unity, is the distance between adjacent farmsteads.

If the point M is taken as a fixed origin, the 60-degree sector YMX in Figure 1 represents an area whose features are repeated on the plain every 60 degrees around M. This sector consists of two 30-degree subsectors YMZ and XMZ, which are mirror images of each other. Thus the point K, with coordinates (5, 2), is equal to the point J in its distance from the origin at M. From the point of view of efficiently enumerating all possible Löschian numbers, the constraint that x is less than or equal to y serves to eliminate duplication of values by confining the enumeration to the subsector YMZ. On the other hand, clarification of the concrete geometrical features of the Löschian landscape requires that the existence of both subsectors be recognized. Keeping the supplier at M fixed, J and K are clearly alternative and competing locations for an adjacent supplier of the good for which $N = 39$. A similar choice exists for any other location of J not lying in the lines MY and MZ. The vexing question of how to choose between these alternatives will be fully answered as the discussion proceeds.

Since the market areas for the suppliers of any given good are identical hexagons, the suppliers themselves form a regular triangular lattice. Just as a farm is the Dirichlet region of its farmstead, so a market area is the Dirichlet region of the supplier at its center. In this paper each lattice of suppliers is denoted by $L(N)$ where N, as noted above, is the number of farms served by each supplier in the lattice. In Figure 1, for example, the points M and J define one possible set of locations for $L(39)$.

The Superimposition of Market Area Nets

Except for the case of $N = 1$ (taken to represent self-sufficiency at each farmstead), every Löschian number identifies a permissible market area hexagon which in turn defines the mesh size of a whole network of market areas covering the entire plain. Lösch builds up his economic landscape by assuming that each geometrically permissible tessellation of market areas is appropriate for the supplying of one particular good. Without loss of generality, the set of suppliers for any such good could be replaced by a set of identical multiple-good firms, but single-good firms are retained here for the sake of simplicity. Although there is an infinity of Löschian numbers, the real world does not contain an infinity of different goods, and Lösch [*11*, p. 135] assumes that the number of different market area tessellations required for the model is large but finite.

One lattice point in $L(1)$ is arbitrarily selected to be the center of one cell in every tessellation to be used; this point becomes by definition a metropolis supplying every good in the economy. The various market area nets are then superimposed on the plain in such a way that each is centered on the chosen metropolitan point [*11*, pp. 124-30].

It should be noted that it is indeed an assumption to regard each possible market area as appropriate for just one good. One can conceive of a situation in which certain market areas are suitable for several goods, whereas others are not suitable for any goods at all. Lösch was aware of these possibilities, but he gave pride of place to a model in which each permissible hexagonal tessellation is used once and once only.

As successive market area nets are laid down, a regular pattern of central places emerges by virtue of the fact that lattices of suppliers of different goods coincide in space on the plain. It is the precise nature of this spatial coincidence that has proved so elusive in the past. The prevailing view, recently expressed by Tarrant [*18*], is that the solution involves rotation of the nets about the central metropolis under two simultaneous constraints: first, that the coincidence of suppliers be maximized, and second that the finished product exhibit alternating city-rich and city-poor sectors. For reasons that will become obvious, let the question of city-rich and city-poor sectors be set aside until the first constraint has been carefully examined.

At this juncture it is appropriate to introduce what Dacey [*5*, p. 116] terms the "primitive function points" of a given lattice $L(N)$—the six points in $L(N)$ that lie closest to the central metropolis. Since the central metropolis and any one primitive function point of $L(N)$ completely define the set of locations occupied by $L(N)$ on the plain, attention may be focused on maximizing the coincidence of the primitive function points of successive lattices within a single 60-degree sector of the economic landscape. Figure 2, details of which are now described, depicts one possible result. For convenience, let $P(N)$ denote a primitive function point of the lattice $L(N)$.

Beginning with the smallest market areas, the first primitive function points entered on Figure 2 are $P(3)$ and $P(4)$, the coordinates of which are respectively (1, 1) and (0, 2). In both cases only one answer is possible. The point at (1, 1) is its own mirror image, and the placement of $P(4)$ at (0, 2) means that $L(4)$ also has a primitive function point at (2, 0). In short, no rotation is possible for lattices $L(3)$ and $L(4)$. Generalizing, it appears from Figure 2 and equation (1) that no rotation is possible in the case of any lattice for which N is of the form x^2 or $3x^2$ (where x is some integer). Actually this rule is not valid, but nothing is lost by temporarily allowing it to stand. Cases in which the rule does not hold are examined below.

The third primitive function point is $P(7)$, which may be at (1, 2) or at (2, 1). The choice, as will be justified below, is entirely arbitrary, and Figure 2 shows $P(7)$ at (1, 2). Note in passing that 7 is a prime number of the form $6k + 1$, where k is some integer.

Next in sequence come $P(9)$ and $P(12)$, for which rotation is impossible. $P(9)$ lies at (0, 3) and $P(12)$ at (2, 2). Both of these locations are necessarily in $L(3)$, and $P(12)$ is also necessarily in $L(4)$. These facts, as will be seen, are highly significant.

It would be tedious to continue this detailed presentation for very

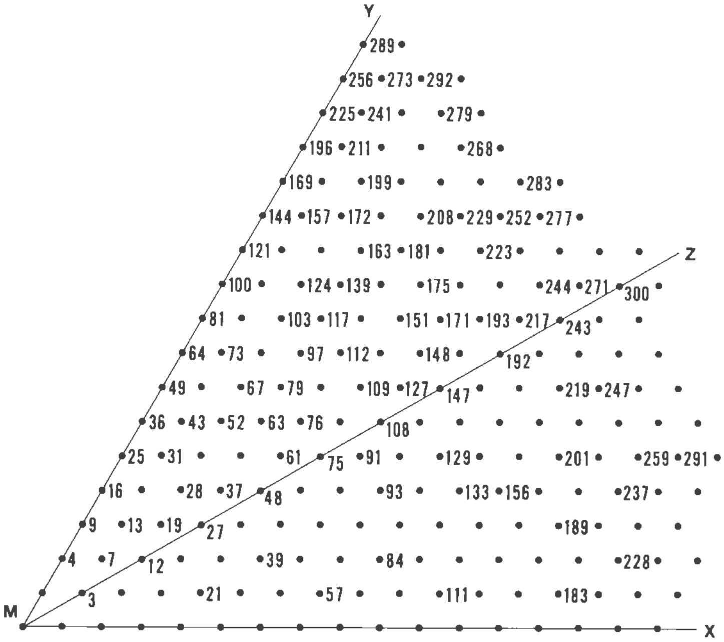

FIG. 2. One Possible Arrangement of Primitive Function Points Up to $P(300)$. Dots are farmstead locations, and M is the central metropolis. Each number shows the location of the primitive function point of the lattice of that magnitude. Numbers lie to the right of the points to which they refer.

long, but it must be carried a little further before all the essential features of the procedure can be distilled. The introduction of $P(21)$, in fact introduces a new element. Here there is apparently a choice between (1, 4) and (4, 1), but the earlier placement of $P(7)$ at (1, 2) means that (4, 1) lies in $L(7)$ whereas (1, 4) does not. In the interest of maximizing the coincidence of lattices, $P(21)$ is therefore placed at (4, 1). Although $P(21)$ does not lie along the lines MY or MZ in Figure 2, 21 is not a prime but a composite number with 3 and 7 as its prime factors. Note that (4, 1) and (1, 4), the two possible locations for $P(21)$, *both* lie in $L(3)$. With $P(7)$ at (1, 2), as noted, $L(7)$ includes (4, 1) but not (1, 4); it perhaps does not come as a surprise to learn that $L(7)$ would include (1, 4), but not (4, 1), if $P(7)$ were placed at (2, 1). This illustrates a vital principle. In cases where $P(N)$ does not lie along MY or MZ (the subsector boundaries), the crucial question is whether or not N is prime. If N is prime, $P(N)$ may be placed above or below MZ at will; and in all such cases, though this is an incidental fact, N is a prime of the form $6k + 1$. If N is composite, the placement of $P(N)$ is predetermined by the choice(s) made for the prime(s) of the form $6k + 1$ by which N is divisible.

The smallest Löschian number divisible by two primes of the form

TABLE 1

CONCORDANCE OF $L(7)$, $L(13)$, AND $L(91)$

Grid Coordinates of Independent Lattices $P(7)$	$P(13)$	Grid Coordinates of Dependent Lattice $P(91)$
1, 2	1, 3	6, 5
1, 2	3, 1	1, 9
2, 1	1, 3	9, 1
2, 1	3, 1	5, 6

Source: calculated by author.

$6k + 1$ is 91, the product of 7 and 13. $P(91)$ has *four* possible locations, namely (1, 9), (5, 6), (6, 5), and (9, 1). There are also *four* distinct combinations for the disposition of $P(7)$ and $P(13)$, each of which has two possible locations. Table 1 shows that there is a one-to-one concordance between these two foursomes. In Figure 2, $P(91)$ is shown at (6, 5), the only location which falls in both $L(7)$ and $L(13)$ with $P(7)$ at (1, 2) and $P(13)$ at (1, 3).

Given, for example, $P(7)$ at (1, 2), the question of whether or not $L(7)$ includes (6, 5), or any other point, can of course be answered graphically. Questions of this type, however, can also be answered rapidly by numerical calculation. The method requires a knowledge of the arithmetic of residues.

Continuing the construction of Figure 2, $P(25)$ and $P(27)$ lie on MY and MZ respectively and involve no rotation. $P(28)$ is predetermined by $P(7)$, and must be at (2, 4) rather than (4, 2). $P(31)$ involves an arbitrary choice between (1, 5) and (5, 1). And so the process continues.

Tarrant [*18*, pp. 116-17] uses the term "symmetric" to refer to lattices whose primitive function points lie along MY and MZ in Figure 2; other lattices are termed "asymmetric." Actually all lattices are equally symmetrical about M, but the terminology is not the point at issue. Tarrant goes on to divide the asymmetric lattices into two types. First, there are those for which N is not divisible by the N-value of any smaller asymmetric lattice; these are the same as lattices for which N is a prime of the form $6k + 1$. Secondly, there are those for which N is a multiple of the N-value(s) of one or more smaller asymmetric lattices; these are the same as lattices for which N is not prime and for which $P(N)$ does not lie on MY or MZ. Tarrant then claims that the primitive function points of the first type of asymmetric lattice must all lie in the same subsector (that is, all above or all below MZ in Figure 2), and he further claims that the primitive function points of the second type of lattice obey an "even or odd" rule as follows: if N is an even multiple of a smaller asymmetric N-value, $P(N)$ must lie in the same subsector as the primitive function point of the smaller lattice, and in the opposite subsector if N is an odd multiple. Both of these claims are false. For lattices in which N is a prime of the form $6k + 1$, the choice between subsectors is in fact completely independent in every case (see Table 1; also Tables 2 and 3, explained below). The second

TABLE 2

CONCORDANCE OF LATTICES DEPENDENT ON $L(7)$, $L(13)$, AND $L(19)$

Grid Coordinates of Independent Lattices			Grid Coordinates of Dependent Lattices			
P(7)	P(13)	P(19)	P(91)	P(133)	P(247)	P(1729)
1, 2	1, 3	2, 3	6, 5	9, 4	11, 7	40, 3
1, 2	1, 3	3, 2	6, 5	11, 1	14, 3	8, 37
1, 2	3, 1	2, 3	1, 9	9, 4	3, 14	23, 25
2, 1	1, 3	2, 3	9, 1	1, 11	11, 7	15, 32
1, 2	3, 1	3, 2	1, 9	11, 1	7, 11	32, 15
2, 1	1, 3	3, 2	9, 1	4, 9	14, 3	25, 23
2, 1	3, 1	2, 3	5, 6	1, 11	3, 14	37, 8
2, 1	3, 1	3, 2	5, 6	4, 9	7, 11	3, 40

Source: calculated by author.

claim is refuted by numerous examples beginning with $L(84)$: 84 is an *even* multiple of 7, but $P(84)$ must go in the *opposite* subsector to $P(7)$ in order to maximize the coincidence of these two lattices. Figure 2 shows $P(7)$ at (1, 2) and $P(84)$ accordingly at (8, 2) rather than (2, 8). Similarly, the number 175 is an *odd* multiple of 7, but $P(175)$ must go in the *same* subsector as $P(7)$ to maximize coincidence. Other examples contradicting Tarrant's rule include $P(156)$, $P(228)$, $P(325)$, and $P(847)$.

The freedom of choice that obtains when N is a prime of the form $6k + 1$ has already been illustrated on a small scale by Table 1. A more complex example, involving the products of the primes 7, 13, and 19, is summarized in Table 2. Each of $P(91)$, $P(133)$, and $P(247)$ has four possible locations. In every case, each of the four possible locations gets used twice in accommodating the eight possible positional combinations of $P(7)$, $P(13)$, and $P(19)$, but the pattern is such that no two sets of locations for $P(91)$, $P(133)$, and $P(247)$ are alike. Finally, $P(1729)$ has eight possible locations, and these eight locations bear a precise one-to-one relationship with the eight positional combinations of $P(7)$, $P(13)$, and $P(19)$, *and* with the eight unique sets of locations for $P(91)$, $P(133)$, and $P(247)$. In short, the positions of the four dependent lattices are uniquely determined once the positions of the three independent lattices have been chosen.

Table 3 gives one further example based on the first four primes of the form $6k + 1$. In this instance, the six lattices involving pairwise combinations of these four primes have been omitted to keep the table of manageable size. Of the five dependent lattices shown, the first four each involve three of the four primes, and their locations are governed, in the manner of Table 2, by the positions adopted for the primes in question. Finally, $P(53599)$ has no fewer than sixteen possible locations, each of which is associated with one, and only one, of the sixteen positional combinations of $P(7)$, $P(13)$, $P(19)$, and $P(31)$. Quite apart from its significance in relation to the construction of the Löschian landscape, the geometrical concordance summarized in these tables is a thing of great beauty.

TABLE 3

CONCORDANCE OF LATTICES DEPENDENT ON $L(7)$, $L(13)$, $L(19)$, AND $L(31)$

	Grid Coordinates of Independent Lattices			Grid Coordinates of Dependent Lattices				
P(7)	P(13)	P(19)	P(31)	P(1729)	P(2821)	P(4123)	P(7657)	P(53599)
1, 2	1, 3	2, 3	1, 5	40, 3	41, 19	58, 11	73, 24	25, 218
2, 1	1, 3	2, 3	1, 5	15, 32	4, 51	17, 54	73, 24	122, 145
1, 2	3, 1	2, 3	1, 5	23, 25	15, 44	58, 11	32, 67	163, 102
1, 2	1, 3	3, 2	1, 5	8, 37	41, 19	6, 61	87, 1	85, 177
1, 2	1, 3	2, 3	5, 1	40, 3	25, 36	41, 33	48, 53	197, 58
2, 1	3, 1	2, 3	1, 5	37, 8	36, 25	17, 54	32, 67	230, 3
2, 1	1, 3	3, 2	1, 5	25, 23	4, 51	33, 41	87, 1	173, 90
2, 1	1, 3	2, 3	5, 1	15, 32	44, 15	61, 6	48, 53	43, 207
1, 2	3, 1	3, 2	1, 5	32, 15	15, 44	6, 61	53, 48	207, 43
1, 2	3, 1	2, 3	5, 1	23, 25	51, 4	41, 33	1, 87	90, 173
1, 2	1, 3	3, 2	5, 1	8, 37	25, 36	54, 17	67, 32	3, 230
2, 1	3, 1	3, 2	1, 5	3, 40	36, 25	33, 41	53, 48	58, 197
2, 1	3, 1	2, 3	5, 1	37, 8	19, 41	61, 6	1, 87	177, 85
2, 1	1, 3	3, 2	5, 1	25, 23	44, 15	11, 58	67, 32	102, 163
1, 2	3, 1	3, 2	5, 1	32, 15	51, 4	54, 17	24, 73	145, 122
2, 1	3, 1	3, 2	5, 1	3, 40	19, 41	11, 58	24, 73	218, 25

Source: calculated by author.

The important practical point revealed by the tables is not that the positioning of the dependent lattices is rigidly controlled, but rather that *there is always a solution,* regardless of the positions selected for the independent lattices representing primes of the form $6k + 1$. Even if the solution in any particular case is unique, it nevertheless exists, and therefore all primitive function points where N is a prime of the form $6k + 1$ may be positioned completely independently of one another. It is this independence which invalidates Tarrant's claim that all such points must be placed in the same subsector.

The existence of alternative positions, within a single 30-degree subsector, for the primitive function points of certain lattices is a matter of some interest in its own right. The smallest lattice of this type is $L(49)$, for which two positions are possible: $P(49)$ may be at (0, 7) or at (3, 5). These two positions represent unequivalent solutions of the Diophantine equation

$$N = x^2 + xy + y^2 .$$

By transposition of the terms x and y, the equivalent solutions (7, 0) and (5, 3) are obtained. The smallest lattices having three, four, and five unequivalent solutions respectively are $L(637)$, $L(1729)$, and $L(8281)$. The lattice $L(53599)$, featured in Table 3, has eight unequivalent solutions and thus sixteen possible positions for its primitive function point within a 60-degree sector. The occurrence of unequivalent solutions is by no means a rarity, for 30.4 percent of all lattices up to $L(10000)$ involve two or more unequivalent solutions. The existence of unequivalent solutions to Diophantine equations is a topic of interest in number theory, and the question of unequivalent solutions in the case of Löschian lattices may be answered by referring to Bolker's rule [*2*, pp. 114–21].

It sometimes occurs that a lattice with two or more unequivalent solutions has one solution of the form $(0, n)$ or (n, n). The lattices $L(49)$ and $L(147)$ are cases in point: $P(49)$ may be at $(0, 7)$ or $(3, 5)$, and $P(147)$ may be at $(7, 7)$ or $(2, 11)$. It is these cases, and others like them, which deny validity to the apparent rule that rotation is impossible for any lattice in which N is of the form x^2 or $3x^2$. In all such cases, however, maximization of coincidence with other lattices is guaranteed if a solution of the form $(0, n)$ or (n, n) is adopted. For the sake of simplicity, therefore, all $P(N)$ where N is of the form x^2 may be taken as lying on MY in Figure 2, and all $P(N)$ where N is of the form $3x^2$ may be taken as lying on MZ.

When N is not of the form x^2 or $3x^2$, $P(N)$ does not lie on MY or MZ, and the important question then is whether or not N is prime. If N is prime, $P(N)$ may be placed either above or below MZ at will: as the tables illustrate, the existence of unequivalent solutions for many lattices provides just the kind of flexibility that Tarrant denies. If N is not prime, then maximization of coincidence of lattices requires that the subsector containing $P(N)$ be determined by the prior placement of primes by which N is divisible.

With maximum coincidence of lattices achieved, the Löschian landscape obeys the following simple rules:

(A) The points of coincidence of any set of lattices $L(a), L(b), L(c), \ldots, L(j)$ define the lattice $L(t)$ where t is the least common multiple of $a, b, c, \ldots, j$.

(B) A farmstead location which is a lattice point in $L(y)$ is also a lattice point in all $L(x)$ where x is any Löschian number that exactly divides y.

To take a simple example, the first rule states that $L(3)$ and $L(4)$ coincide on the plain to form $L(12)$, 12 being the least common multiple of 3 and 4. The second rule states that $L(12)$ exists *only* as the points of coincidence of $L(3)$ and $L(4)$. In like fashion, $L(3)$, $L(4)$, and $L(9)$ coincide to form $L(36)$, but each point in $L(36)$ is also necessarily a point in $L(12)$. Similarily, $L(7)$ and $L(39)$ coincide to form $L(273)$, each point in which is also a point in $L(3)$, $L(13)$, $L(21)$, and $L(91)$. These two rules represent a concrete application of the fact that the least common multiple of any two or more Löschian numbers is always itself a Löschian number [*12*].

Finally, it must be stressed that Figure 2 shows only one of the many possible ways in which the 92 primitive function points up to $P(300)$ may be arranged. Alternative arrangements arise because each lattice in which N is a prime of the form $6k + 1$ involves an arbitrary choice between the upper and lower subsectors. (And this disregards, as noted above, the fact that unequivalent solutions do exist for certain cases in which N is of the form x^2 or $3x^2$.) There are 28 primes of the form

$6k + 1$ up to $P(300)$, and hence there are a staggering 2^{28} distinct ways in which Figure 2 may be drawn. The drawing given here lies at one end of this multitude of possibilities, for all primes of the form $6k + 1$ have been placed in the upper subsector. The only primitive function points appearing in the lower subsector of Figure 2 are those which *must* be there in order to maximize the coincidence of lattices.

City-Rich and City-Poor Sectors

The matter of city-rich and city-poor sectors is best approached by first considering a single lattice $L(N)$ where N is any Löschian number. Let $L(N)$ be centered on the chosen central metropolis in the usual way. Since $L(N)$ is a perfect triangular lattice—that is, the density of points is completely constant throughout the extent of the lattice—it is obvious that the proportion of farmstead points that are members of $L(N)$ is constant in *all* 30-degree sectors radiating from the central metropolis, regardless of their orientation. Now consider two lattices $L(i)$ and $L(j)$. Placing these two lattices together on the plain creates three types of central place: first, those supplying only the good for which $N = i$; second, those supplying only the good for which $N = j$; and third, those supplying *both* of these goods. From what has been said in the previous section, it is evident that the central places supplying *both* goods form the perfect triangular lattice $L(k)$, where k is the least common multiple of i and j. For any 30-degree sector, the proportion of farmstead points that are members of $L(k)$ is therefore constant, and it follows that the same is true for central places supplying each good singly. Extension of this argument by the addition of further lattices leads immediately to the conclusion that every possible 30-degree sector contains exactly the same assortment of combinations of suppliers of different goods. Hence there are no city-rich and city-poor sectors.

The following remarks may serve to clarify this somewhat surprising conclusion. Given a specific combination of goods, central places supplying precisely these goods almost never form a complete triangular lattice in their own right. For illustration, consider the simple case of a two-good economy using lattices $L(7)$ and $L(13)$. The choice of $L(7)$ and $L(13)$ instead of, say, $L(3)$ and $L(4)$ is deliberate, for it is the behavior of the so-called asymmetric lattices that is generally held to give rise to city-rich and city-poor sectors. Let $G(7)$ denote the set of central places supplying *only* the good that uses lattice $L(7)$, and let $G(13)$ denote the set of central places supplying *only* the good that uses lattice $L(13)$; also, let $G(7, 13)$ denote the set of central places supplying *both* goods. It is known that the points of coincidence of $L(7)$ and $L(13)$ create a lattice having the form $L(91)$, and therefore, given that the economy is limited to two goods, the $G(7, 13)$ set of central places is certainly a perfect triangular lattice. On the other hand, neither the $G(7)$ places nor the $G(13)$ places form a complete lattice in their own right. The $G(7)$ places, for instance, have the pattern of an $L(7)$ lattice disfigured by "holes" corresponding to the occurrence of the $L(91)$ lattice of $G(7, 13)$ places. The same holes destroy the perfection of the pattern formed by the $G(13)$ places. The important feature, however, is that *the holes themselves form a perfect lattice.* Hence all 30-degree sectors randomly positioned around the central metropolis contain identical proportions of $G(7, 13)$ places, $G(7)$ places, and $G(13)$ places. No sectoral differentiation exists.

In this simple example the metropolis provides only two goods, and all $G(7, 13)$ places are similarly equipped. In short each $G(7, 13)$ place is a metropolis. Putting this another way, the only set of identically endowed places forming a complete lattice is the set of metropolises: that is, those places which provide every good in the economy. This statement remains true no matter how many different goods the economy is assumed to contain. Below the metropolitan level, each specific combination of goods short of the full array is supplied by a set of $G(i, j, \ldots, n)$ places which do not form a complete triangular lattice. Nevertheless, the spatial pattern formed by each set of $G(i, j, \ldots, n)$ nonmetropolitan places is radially symmetrical with respect to each metropolis. Both (a) the proportion of farmstead locations which become central places, and (b) the proportion of central places offering any given combination of goods, are constant in all 30-degree sectors.

One escape hatch remains to be sealed. Up to this point, the discussion has proceeded on the understanding that the coincidence of lattices has been maximized. Can city-rich and city-poor sectors be made to appear if maximizing the coincidence of lattices is deliberately violated? The answer is no. For illustration, consider the coincidence of $L(7)$ and $L(49)$. Figure 2 maximizes the coincidence of these two lattices by placing $P(7)$ at (1, 2) and $P(49)$ at (0, 7). But suppose that $P(49)$ were perversely placed at (3, 5) instead of (0, 7). With $P(7)$ at (1, 2), the lattice $L(7)$ does not include (3, 5), and thus the maximization of coincidence has clearly been violated. The question naturally arises as to where $L(7)$ and $L(49)$ will now coincide. It turns out that coincidence occurs at (14, 7), this point incidentally being one of the four possible locations for $P(343)$. Returning to the notion of a two-good economy, it is seen that $L(7)$ and $L(49)$ give rise, as in the earlier example, to three types of central place, namely $G(7)$, $G(49)$, and $G(7, 49)$. Since maximization of coincidence has deliberately been avoided, the $G(7, 49)$ places are distributed in the form of $L(343)$ rather than the form of $L(49)$; but nevertheless *they still form a complete triangular lattice.* In general, whether or not the coincidence of lattices is maximized, the points of coincidence of any two lattices, and hence the points of coincidence of *any number* of lattices, themselves form a perfect triangular lattice. The argument of the earlier part of this section therefore remains valid. City-rich and city-poor sectors do not exist whether the coincidence of lattices is maximized or not.

Denial of the existence of alternating city-rich and city-poor sectors in the Löschian landscape will come as a shock to many readers. The supposed sectors are very much an established part of the Löschian litany, and are faithfully described both in introductory texts [*6,* p.

292; *10,* pp. 250-51; *20,* pp. 206-8] and in advanced studies [*5,* p. 121; *15,* p. 572; *16,* pp. 67-69]. Nevertheless the argument presented above seems incontrovertible. Diagrammatic corroboration is easy enough to produce, but unfortunately lacks clarity at any size capable of reproduction on the printed page. Doubters are urged to glue together a few square yards of isometric graph paper and set to work.

It is tempting to speculate on the reasons why Lösch wrongly believed in the existence of city-rich and city-poor sectors. One possibility is that he concentrated his attention too strongly on the area in the immediate vicinity of the central metropolis. In this restricted zone an apparently significant difference between sectors can be made to appear by appropriate positioning of primitive function points. Dacey [*5,* p. 121] seems to make the same error when he writes that "Lösch obtains city-rich and city-poor sectors simply by forcing all primitive function points to cluster within 30-degree sectors." Certainly one may arrange things so that all primitive function points fall in the same set of alternate 30-degree sectors, though in some instances, such as $P(21)$, $P(39)$, and $P(57)$, this will violate the maximization of the coincidence of lattices (see Figure 2). Even if this is done, however, the proportion of farmstead locations supplying any given combination of goods will still be uniform in all sectors. The primitive function points, in short, are merely the tips of icebergs.

In the past, the alternating city-rich and city-poor sectors in the Löschian landscape have been used to account for variations in density reported to occur around such cities as Indianapolis [*13*] and Toledo [*11,* pp. 125, 438-39]. It is now apparent that some other source of explanation must be found. Parr [*14,* p. 206], although he does not question the existence of city-rich and city-poor sectors in the model, wisely suggests that the occurrence of sectoral variations in development around real cities is "probably related to the long-term advantages conferred upon locations along or near intermetropolitan arteries." The corridor theory developed by Russwurm [*17*] and Whebell [*19*] gives support to this view.

Conclusion

The aim of this paper has been to clarify the manner in which the Löschian economic landscape is constructed. Construction has hitherto been thought to proceed under two simultaneous constraints: first, that coincidence of suppliers be maximized, and secondly that the finished landscape exhibit alternating city-rich and city-poor sectors. It is now seen that the second constraint is false. City-rich and city-poor sectors cannot be made to appear whether the coincidence of suppliers is maximized or not. Construction is therefore free to proceed under the single constraint that coincidence of suppliers be maximized.

Each Löschian number N represents a permissible market area which forms the basis of a lattice $L(N)$ of identical suppliers. Systematic construction of the landscape is most readily achieved by concentrating attention on the placement of each lattice's primitive function point $P(N)$ within a single 60-degree sector around the arbitrary central metropolis M (see Figure 2). If N is of the form x^2 or $3x^2$, $P(N)$ is located along the boundary of a 30-degree subsector (that is, along MY or MZ in Figure 2); for these lattices, rotation is usually impossible and never necessary. Cases in which N is not of the form x^2 or $3x^2$ are of two types. In the first type, N is a prime of the form $6k + 1$, and $P(N)$ may be located in either of the two 30-degree subsectors at will. In the second type, N is a composite number, and the choice of subsector for $P(N)$ is governed by the prior choices made for all primes of the first type by which N is divisible (as illustrated by Tables 1, 2, and 3). Construction of the landscape according to these precepts ensures that maximum coincidence of suppliers is achieved.

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