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CHRISTALLERIAN NETWORKS IN THE LÖSCHIAN ECONOMIC LANDSCAPE

John U. Marshall
York University

THE aim of this paper is to fill a small but troublesome gap in the literature of central place theory by demonstrating the exact geometric relationship between the models of Christaller and Lösch. The gap is troublesome not only because it is an intellectual loose end but also because it often leads to confusion in the minds of students. Hudson has examined the relationship between Christaller and Lösch in algebraic terms, and the geometric approach adopted here may be viewed as complementing his work *(1)*.

BACKGROUND. In Lösch's version of central place theory the smallest settlements on the isotropic plain are explicitly assumed to be farms that are hexagonal in shape. Some later writers have taken the basic settlements to be villages rather than farms, but Lösch's position was quite unambiguous: "Let *a* be the distance between the smallest settlements A_1, A_2, and so on, which we have assumed to be farms. Again the most suitable shape of their area is that of the regular hexagon" *(2)*. The farmsteads are located at the centers of the farms and form a regular triangular lattice that serves as a framework for the creation of a network of urban settlements. Only farmstead locations are permitted to become central places. Lösch briefly considered the possibility of locating central places at the midpoints of the triangles formed by the farmsteads, but he rejected this possibility in favor of location solely at farmstead sites *(3)*.

The size of each market area in the Löschian landscape is conventionally measured by the number of farms it contains *(4)*. Since farms and market areas are both represented by regular hexagons, each market area includes some whole farms and some fractions of farms; but in every permissible case the fractions sum to a whole number, and hence the size of each market area is always expressible as an integer. The complete set of these integers, here denoted by Q, is termed the Löschian numbers. As is well known, the Löschian numbers can be generated systematically by the function

$$Q = x^2 + xy + y^2,$$

where $x \leqslant y$ and both x and y are nonnegative integers *(5)*. In what follows, individual Löschian numbers are denoted by N.

The theories of Lösch and Christaller give widely differing emphasis to the principle of the agglomeration of firms. In Lösch's theory this principle is incorporated in two ways: first, by the selection of an arbitrarily located metropolis that, by definition, supplies all goods; and secondly, by the rotation of market area nets about this metropolis to maximize the spatial coincidence of suppliers. It is of paramount significance, however, that Lösch performed these acts only after the size of market area adopted by each type of business had been determined in accordance with the assumption that excess profits would be minimized. Lösch's desire to minimize excess profits clearly took precedence over his attempt to bring about the agglomeration of firms. Quite the opposite is true in Christaller's version of the theory. Indeed, it is precisely because Christaller gave priority to the agglomerative principle that his models possess their distinctive hierarchical structuring, a feature not found in the Löschian landscape *(6)*. To the extent that real central place firms have a natural tendency to agglomerate, Christaller's models are more realistic than the Löschian landscape. It is therefore pertinent to ask for an exact specification of the ways in which the Christallerian patterns may be extracted from the complex geometry of Lösch's model.

EXTRACTION OF CHRISTALLERIAN NETWORKS. The simplest Christaller model is the so-called marketing or $k = 3$ model, in which market areas on any level of the hierarchy are three times as large as those on the level immediately below. To derive this model from the Löschian landscape one

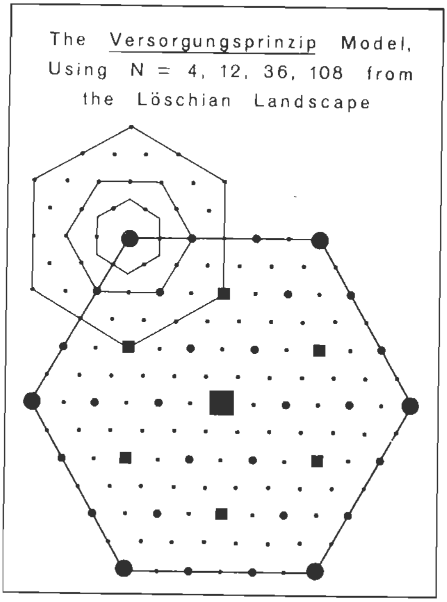

Figure 1. In this and subsequent figures the smallest dots represent farmsteads. Larger symbols represent four hierarchical orders of central places.

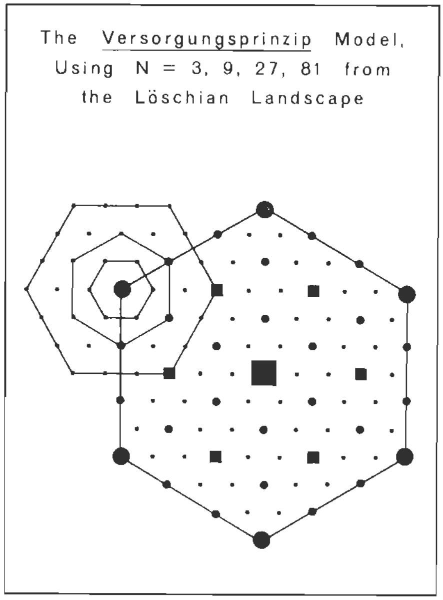

Figure 2.

extracts from the latter a set of market areas having Löschian numbers, *N*, such that each value of *N* is exactly three times its predecessor. A crucial point, overlooked in the literature to date, is that there is an infinity of sets with this property, commencing with the following examples:

N = 3, 9, 27, 81, . . .
N = 4, 12, 36, 108, . . .
N = 7, 21, 63, 189, . . .
N = 13, 39, 117, 351, . . .
N = 16, 48, 144, 432, . . .
N = 19, 57, 171, 513, . . .

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The arrangement of central places of various orders is the same for all members of this series; the difference between the models lies solely in the proportion of farmstead sites that fail to become central places (Figures 1, 2 and 3). The word "arrangement" refers to the positions of the central places in relation to one another. The absolute spacing of places is a different matter, dependent on the distance separating adjacent farmsteads in each particular case. If farms in Figure 1 are presumed large and those in Figure 3 are presumed small, the spacing of each order of places could be greater in the former than in the latter. It needs to be kept in mind that the typical spacings recorded by Christaller for Bavaria were determined empirically *(7)*. No information on spacing is inherent in the models.

In similar fashion Christaller's traffic or $k = 4$ model may be derived by extracting a set of Löschian market areas in which each value of *N* is four times its predecessor. Again there is an infinity of such sets, and again the arrangement of central places is the same in all cases (Figures 4 and 5). The first few sets in this series are as follows:

N = 3, 12, 48, 192, . . .
N = 4, 16, 64, 256, . . .
N = 7, 28, 112, 448, . . .
N = 9, 36, 144, 576, . . .
N = 13, 52, 208, 832, . . .
N = 19, 76, 304, 1216, . . .

Any set of Löschian market areas in which the successive values of *N* are related

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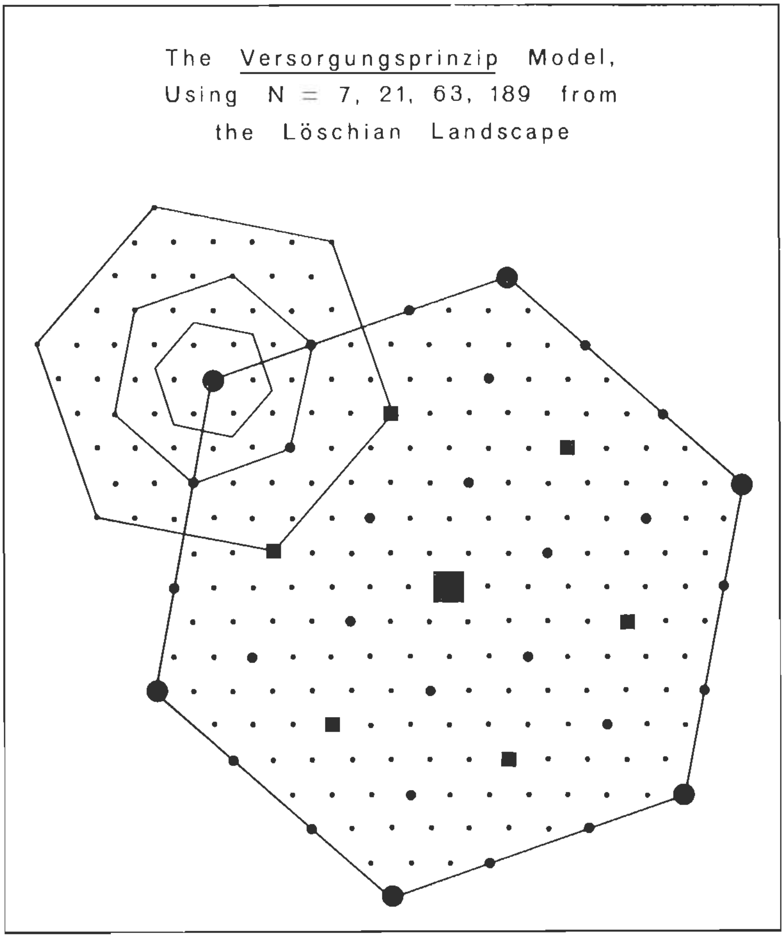

Figure 3.

by a constant multiplier represents a hierarchical system in which the number of centers on any one level is a constant multiple of the number of centers on all higher levels combined. Lösch termed such systems "regions with homogeneous structure," and he used k to denote the multiplier linking successive values of N *(8)*. The symbol k simultaneously identifies the ratio of the sizes of market areas on any two adjacent levels in the hierarchy. Lösch evidently believed that the smallest value of N in any "region with homogeneous structure" had to equal the value of k in that same system, as in Figures ~~1~~ and 5. But it 2
is clear from Figures ~~2~~, 3 and 4 that there 1
need be no such restriction. Previous research has not emphasized the fact that N and k have entirely distinct meanings: N is the number of farms served by a supplier, whereas k is a ratio of market areas sizes. Moreover, it has recently been proved that the product of two Löschian numbers is always itself a Löschian number *(9)*. Two important findings follow. First, k can take any value that is a possible value of N.

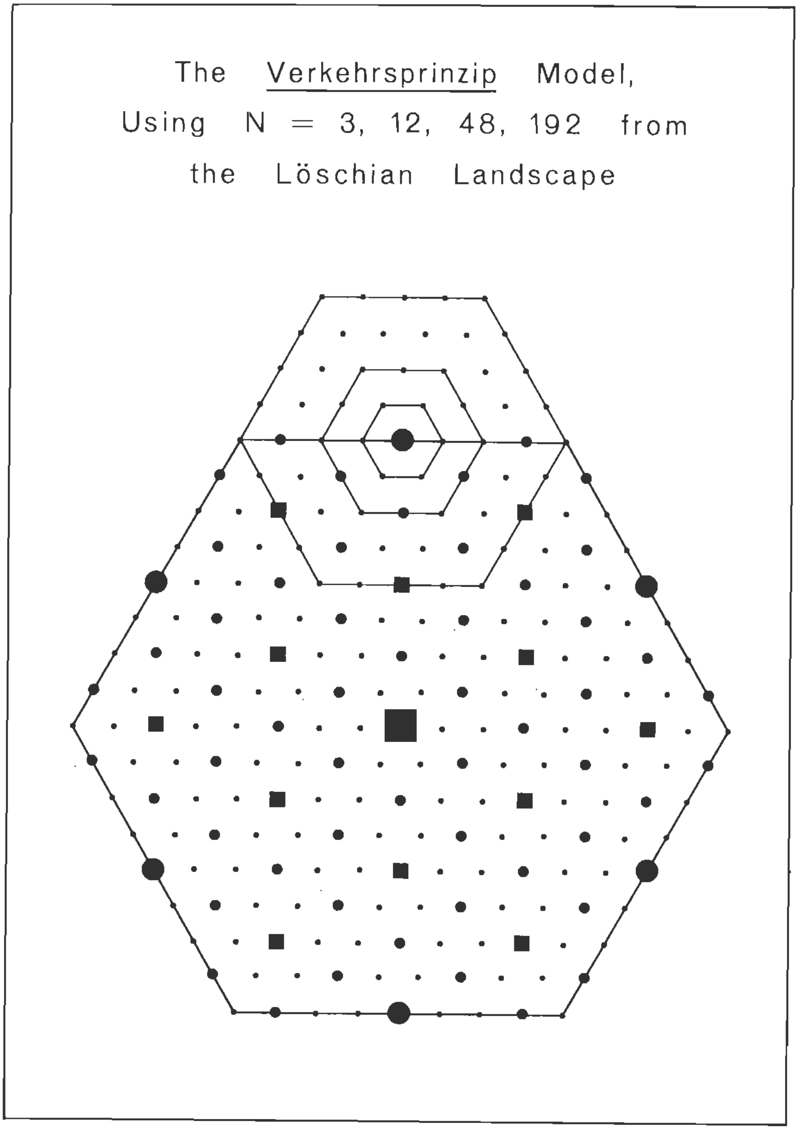

Figure 4.

Secondly, for any value of k, the smallest market area extracted may also have any value of N. The total number of possible "regions with homogeneous structure" is therefore an infinity of infinities.

In addition to the $k = 3$ and $k = 4$ models, Christaller attempted to design a $k = 7$ or "administration" model in which every center would lie wholly inside the market area of one center of the next higher rank. Despite a gallant attempt, he was unable to draw this third model without distorting

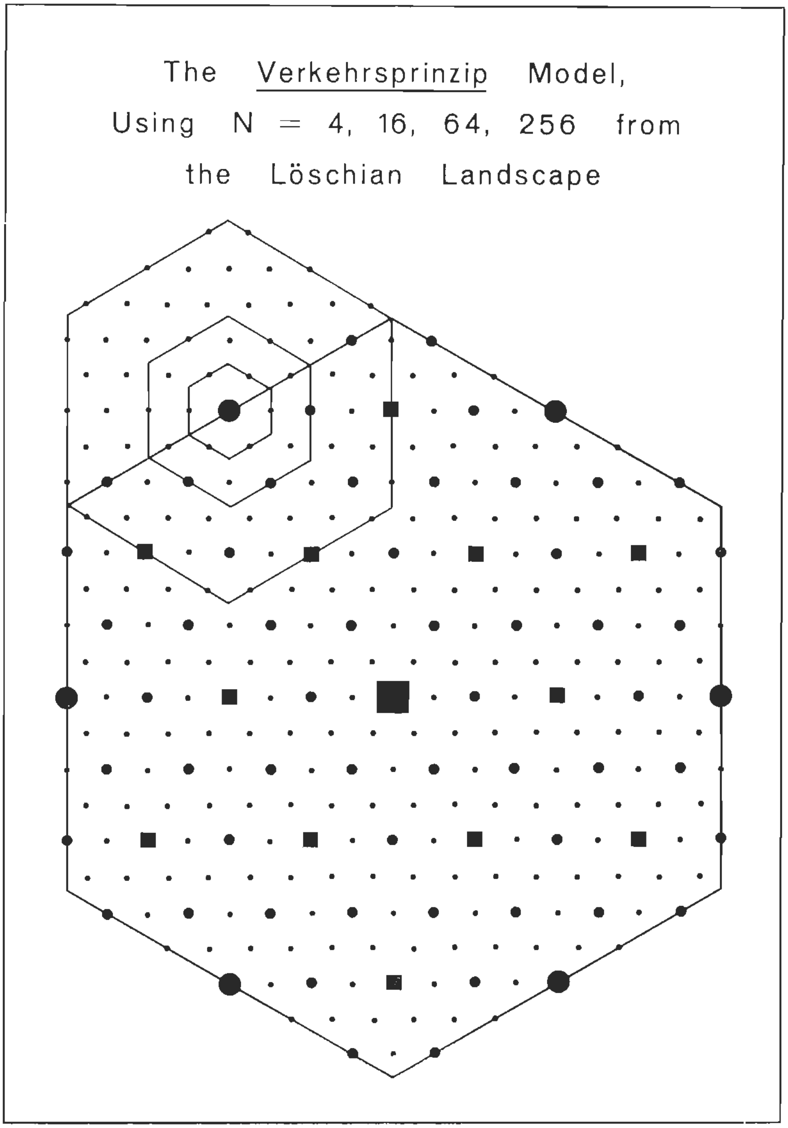

Figure 5.

the regular hexagonal pattern of centers and market areas *(10)*. Lösch pointed out *(11)* that one solution is

$$N = 7, 49, 343, 2401, \ldots$$

Suffice it to say that once again the number of possible solutions is infinite.

7^n

It is possible to extract from the Löschian landscape hierarchical systems which may be termed quasi-Christallerian in the sense that they do not incorporate a constant value of k. For example, k may be allowed to alternate between four and three in generating successive values of N (Fig-

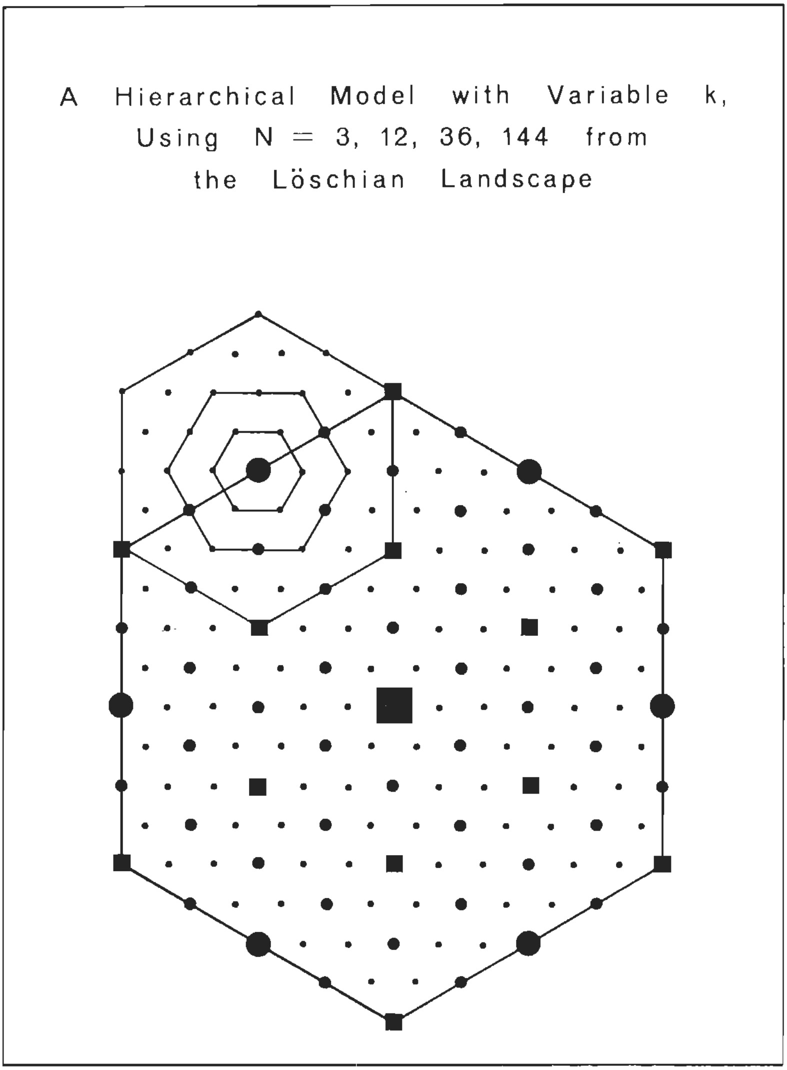

Figure 6.

ure 6). The resulting pattern could occur if the traffic and marketing principles alternated as the various orders of central places were added to the system. Variable-*k* models may have significance for empirical research in view of evidence that the lowest ranking centers (usually termed hamlets) in certain areas are much more numerous than would be predicted on the basis of a fixed-*k* model fitted to the higher orders *(12)*.

Certain hierarchical models seem more likely to have real counterparts than others. For a given maximum value of *N*, the number of different-sized market areas available to suppliers decreases as the average value of k increases. Thus an increase in the value of k implies that activities must be increasingly concentrated in larger numbers in fewer centers. It seems reasonable to assume that real conditions will favor maximization of the available number of different-sized market areas. For in this case, within the limitation set by the agglomerative tendency of firms, the maximum number of viable firms is sustained and excess profits are to some extent held in check, as expected in a competitive economy. It follows that the models most likely to have real counterparts in a competitive economy are those in which the values of k are small. This result is consistent with the suggestion that real entrepreneurs are simply not likely to conceive of a locational strategy that would lead to a pattern with a high value of k. The logic of the $k = 3$ and $k = 4$ models can almost be grasped intuitively, but the same cannot be said for models with higher values of k. Christaller himself was unable to draw even the $k = 7$ model in its ideal geometrical form.

CONCLUSION. The merit of the Löschian landscape, aside from the fact that it is a geometer's delight, is that it suggests ways in which the less complex but more plausible Christallerian theory may be made more responsive to the needs of empirical research. By explicitly considering the number and spacing of farmsteads, and by drawing attention to the fact that the sequence of market area sizes need not be based on a constant multiplier, Löschian theory provides a basis for greatly increased flexibility in the use of hierarchical models of the type originated by Christaller. This note has demonstrated that Christallerian and quasi-Christallerian patterns lie embedded in the Löschian landscape in literally an infinite number of ways. Thus Lösch's central place theory, though quite unrealistic in itself, may enhance our ability to construct models that faithfully reflect the variety of spatial structures found in real central place systems.

* * *

(1) John C. Hudson, "An Algebraic Relation between the Lösch and Christaller Central Place Networks," *The Professional Geographer*, Vol. 19 (1967), pp. 133–135.

(2) August Lösch, *The Economics of Location*, translated by W. H. Woglom and W. F. Stolper (New Haven: Yale University Press, 1954), p. 115.

(3) Lösch, op. cit., pp. 121–122.

(4) Lösch, op. cit., pp. 116–119.

(5) Michael F. Dacey, "A Note on Some Number Properties of a Hexagonal Hierarchical Plane Lattice," *Journal of Regional Science*, Vol. 5, No. 2 (1964), pp. 63–67.

(6) Walter Christaller, *Central Places in Southern Germany*, translated by C. W. Baskin (Engelwood Cliffs, N.J.: Prentice-Hall, Inc., 1966), pp. 58–80; Brian J. L. Berry and William L. Garrison, "Recent Developments of Central Place Theory," *Papers and Proceedings*, Regional Science Association, Vol. 4 (1958), pp. 107–120.

(7) Christaller, op. cit., p. 67.

(8) Lösch, op. cit., p. 131.

(9) John U. Marshall, "The Löschian Numbers as a Problem in Number Theory," *Geographical Analysis*, Vol. 7 (1975), p. 425.

(10) Christaller, op. cit., pp. 78–79.

(11) Lösch, op. cit., p. 132.

(12) John R. Borchert and Russell B. Adams, *Trade Centers and Trade Areas of the Upper Midwest*, Urban Report No. 3 (Minneapolis: Upper Midwest Economic Study, 1963), p. 2; John U. Marshall, *The Location of Service Towns* (Toronto: University of Toronto Press, 1969), p. 140.