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Counting plane trees

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Introduction

Using Polya enumeration and generating functions, we derive formulas to count the number of plane trees, first up to orientation preserving homeomorphism of the plane, and then up to arbitrary homeomorphism. As a corollary we also get a strikingly simple formula for the number of plane trees that possess a bilateral symmetry.

Plane trees up to orientation preserving homeomorphism

Let $p_0, \ldots, p_{2n-1}$ be points equally spaced along the perimeter of a disk in the plane. A *planar n-pairing* is here a collection of n non-intersecting line segments connecting the points p_i in pairs. The cyclic group $G = C_{2n} = Z/\langle 2n \rangle$ acts by rotations on the set $\mathcal{P}$ of planar pairings. The number of orbits of this action, $p(n)$, is just the number of distinct planar n-pairings up to rotation.

A planar n-pairing partitions the disk into $n+1$ cells. Choosing a point in each cell and connecting points in adjacent cells by line segments gives rise to a planar embedded tree with $n+1$ nodes. Accordingly, $p(n)$ also represents number of planar embedded trees with $n+1$ nodes up to orientation preserving homeomorphisms of the plane.

Our first theorem gives a formula for $p(n)$:

Theorem 1

$$p(n) = \frac{\dfrac{\binom{2n}{n}}{n+1} + \left[\dfrac{1}{2}\binom{n+1}{(n+1)/2}\right]_{n \text{ odd}} + \sum_{\substack{d|n \\ d<n}} \phi\left(\frac{n}{d}\right)\binom{2d}{d}}{2n}.$$

Proof The formula follows from an application of Polya's method of enumeration which we now recall. Assign to each element p of $\mathcal{P}$ a weight $w(p)$ equal to the the reciprical of the size of the orbit containing p. The number of orbits is then $\sum_{p \in \mathcal{P}} w(\mathcal{P})$. Let Stab p be the subgroup of G that fixes p. Since the size of an orbit is the index of the stabilizer of any of its elements $w(p) = \frac{|\text{Stab } p|}{|G|}$. Let $Fixg$ be the set of points fixed by g. Then the calculation

$$\sum_{p \in \mathcal{P}} w(\mathcal{P}) = \frac{1}{|G|} \sum_{p \in \mathcal{P}} |\text{Stab } p| = \frac{1}{|G|} \sum_{p \in \mathcal{P}} \sum_{\substack{g \in G \\ gp = p}} 1 = \frac{1}{|G|} \sum_{g \in G} \sum_{\substack{p \in \mathcal{P} \\ gp = p}} 1 = \frac{1}{|G|} \sum_{g \in G} |\text{Fix } g|$$

allows us to replace a sum over $\mathcal{P}$ with a sum over G, which often, as here, turns out to be more tractible.

To apply Polya's method we must compute $|\text{Fix } g|$ for each g in C_{2n}. This will be done in three lemmas.

Lemma 1 If g is the identity, $|\text{Fix } g|$, the total number of planar n-pairings is given by the Catalan number

$$c_n = \frac{\binom{2n}{n}}{n+1} .$$

Proof This is well known, but we offer a quick combinatorial argument for the sake of completeness. First we will associate to each planar n-pairing a rectilinear path through an $n \times n$ grid. These paths start at the upper left corner, end at the lower right corner, and never crossing below the diagonal. Begin at p_0 and proceed clockwise through the $2n$ points, adding a unit segment to the path for each point. If a point represents first contact with a particular line segment, move a unit to the right; if it represents second contact, move down.

The *total* number of paths through an $n \times n$ grid is $\binom{2n}{n}$. We now construct an $n+1$ to 1 from the set of all paths through an $n \times n$ grid to the subset of paths which never pass below the diagonal. Start with an arbitrary path T. Add a right moving segment at the end of T, forming a path T' through an $n \times n+1$ grid. Let t be the unique point on T' farthest to the right of the diagonal of the $n \times n+1$ grid. Construct a new path T'' through an $n \times n+1$ grid, starting at t, by extending T' periodically if necessary. T'' never passes below the diagonal of the $n \times n+1$ grid. Thus, it *must* begin with two right moving segments. Delete the first segment to obtain a path T''', through an $n \times n$ grid, that never passes below the diagonal.

To see that the map is $n+1$ to 1, reverse the process. Adjoin a right moving segment at the start of T''' to recover T''. Then the start of T', and so of T, may be any right endpoint of one of the $n+1$ right moving segments of T''.■

Lemma 2 If n is odd and g is the element of order 2 in C_{2n}, $|\text{Fix } g|$ is

$$\frac{1}{2}\binom{n+1}{(n+1)/2} .$$

Proof Let P be an element of Fix g. Since the number of segments in P is odd, g fixes at least one segment. To fix a segment, g must interchange its endpoints, making the segment a diagonal. But P contains at most one diagonal. In an obvious sense, we can take the quotient of P by g. We may regard the result as a planar $(n-1)/2$-pairing plus a loop coming from the diagonal attached between two point. Let $p_0, \ldots, p_{n-1}$ be points equally spaced along the perimeter of a disk in the plane. There are n ways to choose one of these points for the base of a loop and

$$\frac{\binom{n-1}{(n-1)/2}}{(n+1)/2}$$

ways to form a planar pairing on the remaining points. Finally,

$$n\frac{\binom{n-1}{(n-1)/2}}{(n+1)/2} = \frac{1}{2}\binom{n+1}{(n+1)/2} .\blacksquare$$

Lemma 3 Let e be the order of g. Assume that $e > 1$ and $e|n$ and set

$d = n/e$. Then $|\text{Fix } g|$ is

$$\binom{2d}{d} .$$

Proof As is well known, the generating function for the Catalan numbers

$$f(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots = 1 + x + 2x^2 + 5x^3 + \cdots$$

may be expressed as

$$f(x) = \frac{1 - (1-4x)^{1/2}}{2x}$$

and satisfies the functional equation

$$x f(x)^2 - f(x) + 1 = 0 .$$

These facts are easily derived from Lemma 1; alternatively, the functional equation follows directly from standard combinatorial reasoning.

Suppose P is a planar n-pairing fixed by g. Suppose p_0 is connected to p_i. Then i must be odd so that the points between p_0 and p_i may be paired. Furthermore, either $0 < i < 2n/e = 2d$ or $2n - 2d < i < 2n$, lest the segment that connects p_0 to p_i cross its translate by either g or g^{-1}.

By reflection through the diagonal containing p_0 we see that the number of planar n-pairings fixed by g with $0 < i < 2d$ equals the number with $2n - 2d < i < 2n$. Temporarily assume that $0 < i < 2n/e = 2d$. There are $c_{(i-1)/2}$ ways to pair the points $\{p_1, \ldots, p_{i-1}\}$. The pairing of the points $\{p_1, \ldots, p_{i-1}\}$ determines the pairing of the sets $\{p_{1+2jd}, \ldots, p_{i-1+2jd}\}$ for $j = 0, \ldots, e-1$, by translation. Removing all these points and segments, leaves a planar $2n-(i+1)e$-pairing which is also fixed by an element of order e, but now in $C_{2n-(i+1)e}$.

To formalize the recursion, we introduce the generating function

$$g(x) = b_0 + b_1 x + b_2 x^2 + b_3 x^3 + \cdots$$

where b_j is the number of planar je-pairings fixed by an element of order e. The previous paragraph show that if $j > 0$

$$b_j = 2(c_0 b_{j-1} + c_1 b_{j-2} + \cdots + c_{j-1} b_0) ,$$

where the factor 2 takes account of the pairings with $2n - 2d < i < 2n$. We may express this equation as a generating function identity

$$g(x) = 2xf(x)g(x) + 1$$

or

$$f(x) = \frac{g(x) - 1}{2xg(x)} .$$

Substituting this expression for $f(x)$ into the function equation

$$xf(x)^2 - f(x) + 1 = 0$$

and simplifying yields

$$g(x) = (1 - 4x)^{-1/2}$$

and expanding $g(x)$ as a Taylor series yields the desired result. ■

Putting the lemmas together yields Theorem 1. Lemma 1 contributes the first term in the numerator, Lemma 2 the second. If $e|n$, the number of elements in C_{2n} of order e is $\phi(e)$, so Lemma 3 contributes the third term.■

Comment Perhaps it is worth noting that the generating function $g(x)$ does not depend on e.

Comment The formula may be very slightly simplified:

$$p(n) = \frac{\left[\binom{n}{\lfloor n/2 \rfloor}\right]_{n \text{ odd}} - \binom{2n}{n-1} + \sum_{d|n} \phi\left(\frac{n}{d}\right)\binom{2d}{d}}{2n} .$$

Plane trees up to arbitrary homeomorphism

The dihedral group $G = D_{2n}$ also acts by rotations on the set $\mathcal{P}$ of planar pairings. The number of orbits of this action, $q(n)$, is just the number of distinct planar n-pairings up to rotation and reflection. As before, $q(n)$ also represents number of planar embedded trees with $n + 1$ nodes up to homeomorphisms of the plane. Our second theorem gives a formula for $q(n)$:

Theorem 2

$$q(n) = \frac{\dfrac{\binom{2n}{n}}{n+1} + \left[\binom{n+1}{(n+1)/2}\right]_{n \text{ odd}} + \sum_{\substack{d|n \\ d<n}} \phi\left(\frac{n}{d}\right)\binom{2d}{d} + n\binom{n}{\lfloor n/2 \rfloor}}{4n} .$$

Proof To apply Polya's method it will be sufficient to compute $|\text{Fix } g|$ for the reflections g in D_{2n}, the contribution of the rotations being as in Theorem 1. If g is a reflection, let us orient ourselves so that the axis of g is vertical.

Lemma 4 Let n be odd. Consider reflections g that fix two points from $\{p_0, \ldots, p_{2n-1}\}$, say p_i and p_{i+n}. The sets Fix g are disjoint and the total of $|\text{Fix } g|$ for all such g is

$$\frac{1}{2}\binom{n+1}{(n+1)/2} .$$

Proof Consider a reflection g that fixes the points p_i and p_{i+n}. If P is a planar n-pairing fixed by g there must be a diagonal segment Δ in P connecting p_i and p_{i+n}. Form a new planar n-pairing P' by reflecting the segments to the left of the Δ, as viewed from p_i, across the diagonal perpendicular to Δ. As i varies, the pairings P' will range over all the pairings counted in Lemma 2, so the total of $|\text{Fix } g|$ for all g that fix two points is also

$$\frac{1}{2}\binom{n+1}{(n+1)/2} . \blacksquare$$

Note that if n is even, reflections that fix two points from $\{p_0, \ldots, p_{2n-1}\}$ fix no pairings at all.

Lemma 5 Let g be a reflection with no fixed points in $\{p_0, \ldots, p_{2n-1}\}$. Then

$$|\text{Fix } g| = \binom{n}{\lfloor n/2 \rfloor} .$$

Proof Consider the generating function

$$h(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

where a_n is the number of fixed points of such a $g \in D_{2n}$ on $\mathcal{P}$. We first establish the identity

$$h(x) = \frac{f(x^2)}{1 - xf(x^2)} = f(x^2) + xf(x^2)^2 + x^2 f(x^2)^3 + \cdots .$$

The contribution to $h(x)$ from $x^j f(x^2)^{j+1}$ corresponds to pairings with j horizontal segments crossing the axis of reflection, the factor x^j taking account of these segments. Whether above the highest horizontal segment, between two horizontal segments, or below the lowest, the points on the left side of the axis are paired among themselves, the pairing being mirrored on the right. The Catalan numbers give the number of ways the points on the left side of each of these $j+1$ regions may be paired. This accounts for the factor $f(x^2)^{j+1}$.

By elementary algebra

$$h(x) = \frac{\dfrac{1-(1-4x^2)^{1/2}}{2x^2}}{1 - x\left(\dfrac{1-(1-4x^2)^{1/2}}{2x^2}\right)} = (1-4x^2)^{-1/2}\left(1 + \frac{1}{2x}\right) - \frac{1}{2x}$$

and it follows that

$$h(x) = \binom{0}{0} + \frac{1}{2}\binom{2}{1}x^1 + \binom{2}{1}x^2 + \frac{1}{2}\binom{4}{2}x^3 + \binom{4}{2}x^4 + \frac{1}{2}\binom{6}{3}x^5 + \binom{6}{3}x^6 + \cdots ,$$

as desired.■

Theorem 2 follows on noting that there are n reflections of the type considered in Lemma 5.

Comment The number of plane trees with n nodes which possess at least one bilateral symmetry is

$$2q(n) - p(n) = \frac{\left[\dfrac{1}{2}\dbinom{n+1}{(n+1)/2}\right]_{n \text{ odd}} + n\dbinom{n}{\lfloor n/2 \rfloor}}{2n} = \binom{n-1}{\lfloor (n-1)/2 \rfloor}$$

or equivalently, the number of plane trees with n edges which possess at least one bilateral symmetry is $\binom{n}{\lfloor n/2 \rfloor}$.

Values of $p(n)$ and $q(n)$

The following values were computed using *Mathematica.*

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n	p(n)	q(n)
1	1	1
2	1	1
3	2	2
4	3	3
5	6	6
6	14	12
7	34	27
8	95	65
9	280	175
10	854	490
11	2694	1473
12	8714	4588
13	28640	14782
14	95640	48678
15	323396	163414
16	1105335	555885
17	3813798	1913334
18	13269146	6646728
19	46509358	23278989
20	164107650	82100014
21	582538732	291361744
22	2079165208	1039758962
23	7457847082	3729276257
24	26873059986	13437206032
25	97239032056	48620868106
26	353218528324	176611864312
27	1287658723550	643834562075
28	4709785569184	2354902813742
29	17280039555348	8640039835974
30	63583110959728	31791594259244