K J LMS 31 (1956)

I wish to express my thanks to Prof. T. G. Room for his help during the preparation of this paper, and to acknowledge the support of the Research Committee of Sydney University in the form of a Student Research Grant.

References.

2769

1. Room, *American J. of Math.*, 74 (1952), 967.
2. Cartan, *Actualités Scientifiques et Industrielles* (1938), 701.
3. Todd, *Proc. London Math. Soc.* (2), 30 (1930), 513.

School of Mathematics,
University of Sydney.

REAL QUARTIC FIELDS WITH SMALL DISCRIMINANT

H. J. GODWIN*.

The purpose of this paper is to give a table of the totally real quartic fields with discriminants less than 11664; the method of discovering them is based on one used earlier [2] in connection with the problem of the homogeneous minimum of the product of five real linear forms. When I did this work I was unaware that a table with discriminants up to 8112 had been given by Delone and Faddeev ([1], p. 155) and I am grateful to the referee for informing me of this. My results reveal one error in the table given in [1]: the quartic $x^4-x^3-7x^2+8x-2$ with discriminant 7260 factorizes into $(x^2-3x+1)(x^2+2x-2)$ and so its zeros do not generate quartic fields. In view of the considerable corroboration afforded to my work by [1] it has been thought unnecessary to give the arithmetical details in full, and I give only the working for discriminants less than 2000, which is a sufficiently high value to enable most of the principles involved to be illustrated.

Throughout the paper we use small Latin letters to denote rational integers and small Greek letters to denote algebraic integers. We use K to denote a quartic field: such fields may or may not possess a quadratic subfield k. If K has a quadratic subfield then it is generated by a number $\sqrt{\mu}$ of the form $\sqrt{(p+q\sqrt{m})}$. The discriminant of $K = K(\sqrt{\mu})$ depends on the quadratic character of μ with respect to 4 and divisors of 4 in $k(\sqrt{m})$; for details see Mayer [4] or Sommer ([5], pp. 306-308).

The basis of the method used here is

THEOREM 1. *Let K be a totally real quartic field with discriminant Δ. Then there is at least one polynomial $f(x) = x^4-ax^3+bx^2-cx+d$ with*

* Received 20 September, 1955; read 17 November, 1955.

$S = 3a^2 - 8b \leqslant (32\Delta)^{1/3}$ *such that either* (i) $f(x)$ *is irreducible over the rationals and* K *is generated by one of the zeros of* $f(x)$, *or* (ii) $f(x)$ *is the square of a quadratic* $g(x)$ *whose zeros generate a quadratic subfield of* K.

Proof. The integers of K, with their conjugates, form a four-dimensional lattice with determinant $\sqrt{\Delta}$. Among the points of the lattice are $O(0, 0, 0, 0)$ and $I(1, 1, 1, 1)$; the lattice therefore projects into a three-dimensional lattice with determinant $\frac{1}{2}\sqrt{\Delta}$ in a hyperplane perpendicular to OI. Since the critical determinant of a sphere of radius R is $R^3/\sqrt{2}$ (classical; first proved by Gauss), there is a lattice point $(\theta_1, \theta_2, \theta_3, \theta_4)$ for which

$$\Sigma(\theta_i - \bar{\theta})^2 \leqslant (\sqrt{2} \cdot \tfrac{1}{2}\sqrt{\Delta})^{2/3} \quad [\text{where } \bar{\theta} = \tfrac{1}{4}(\theta_1 + \theta_2 + \theta_3 + \theta_4)].$$

Thus $\theta_1, \theta_2, \theta_3, \theta_4$ are the zeros of a polynomial $f(x) = x^4 - ax^3 + bx^2 - cx + d$ for which

$$S = 3a^2 - 8b = 4\Sigma(\theta_i - \bar{\theta})^2 \leqslant (32\Delta)^{1/3}.$$

Now if θ_1 generates K, then $f(x)$ is irreducible over the rationals and we have (i), but if K has a quadratic subfield k, then θ_1 may belong to k in which case we shall have $\theta_1 = \theta_2$, $\theta_3 = \theta_4$ (on arranging the suffixes appropriately), θ_3 being the conjugate of θ_1 in k. Hence

$$f(x) = (x - \theta_1) \dots (x - \theta_4) = \big((x - \theta_1)(x - \theta_3)\big)^2$$

and we have (ii). This completes the proof of Theorem 1.

Note: If K has no subfield then only (i) can arise, but if K has a subfield then there may be polynomials $f(x)$ of each kind arising.

The result of Theorem 1 is substantially that of [1], p. 147, equation (2). The method of application of it is, however, different in the present paper and to show this we shall set out the working for discriminants less than 2000. We can state the result as

THEOREM 2. *The only numbers less than* 2000 *which are the discriminants of totally real quartic fields are* 725, 1125, 1600 *and* 1957. *There is just one field (apart from conjugate fields) corresponding to each discriminant and, except in the case of discriminant* 1957, *each field has a quadratic subfield.*

Proof. If $\Delta < 2000$, then by Theorem 1, $S < 40$. We thus have to find the polynomials with four real zeros for which $S \leqslant 36$, since $S \equiv 0$, 3 or 4 modulo 8. Since we do not alter K by adding the same rational integer to each zero we may suppose that $f(x)$ has zeros $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ where $\alpha_1 < \alpha_2 < \alpha_3 < \alpha_4$ and $0 < \alpha_1 < 1$. We shall say that $f(x)$ is of type $([\alpha_1], [\alpha_2], [\alpha_3], [\alpha_4])$.

Since we change the signs of the zeros without altering K, we need not consider both types (0, 0, 1, 3) and (0, 2, 3, 3); in the case of a type such as (0, 2, 2, 4) which remains unaltered on replacing α_i by $5-\alpha_i$, we can impose some other condition on grounds of symmetry. Since S is least when the α_i are grouped as closely as possible about their average, we see that in type (0, 1, 1, 5), S is greater than it would be with $\alpha_1 = 1$, $\alpha_2 = \alpha_3 = 2$, $\alpha_4 = 5$, *i.e.* in type (0, 1, 1, 5) we have $S > 36$. Applying these principles generally, we can exclude all but the following types: those with $[\alpha_4] < 3$ and also (0, 0, 0, 3), (0, 0, 1, 3), (0, 0, 2, 3), (0, 0, 3, 3), (0, 1, 1, 3), (0, 1, 2, 3), (0, 0, 0, 4), (0, 0, 1, 4), (0, 0, 2, 4), (0, 0, 3, 4), (0, 1, 1, 4), (0, 1, 2, 4), (0, 1, 3, 4), (0, 2, 2, 4), (0, 1, 2, 5), (0, 1, 3, 5), (0, 2, 2, 5), and (0, 2, 3, 5).

We shall make continual use of the inequalities

$$(I_n) \quad |f(n)| = |(n-\alpha_1)\ldots(n-\alpha_4)| \geqslant 1.$$

I_n is relaxed, a is unaltered and S decreased if, when two or more α_i have the same integral part, we replace them by their average, or, in other words, treat them as equal. We may therefore do this when seeking bounds on a or a lower bound for S.

We now consider the various types in detail; it is convenient to alter the order in which they appear in the list above.

Types (0, 0, 0, 3) *and* (0, 0, 0, 4). Since $|x(x-1)| \leqslant \frac{1}{4}$ in (0, 1) we have $\Pi|\alpha_i(1-\alpha_i)| < (5)(4)(\frac{1}{4})^3 < 1$, in contradiction to I_0, I_1.

Type (0, 0, 1, 3). I_0, I_1 give $0{\cdot}35 < \alpha_1 \ (=\alpha_2) < 0{\cdot}43$, whence I_2 gives $\alpha_3 < 1{\cdot}82$. I_0, I_1 now give $0{\cdot}370 < \alpha_1 < 0{\cdot}363$, which is impossible.

Type (0, 1, 1, 3). I_1, I_2 give $1{\cdot}57 < \alpha_2 < 1{\cdot}50$. Impossible.

Type (0, 0, 1, 4). I_1 gives $\alpha_1 < \frac{1}{2}$ whence $S > 36$ if $\alpha_4 > 4{\cdot}2$. If $\alpha_4 \leqslant 4{\cdot}2$ then $0{\cdot}34 < \alpha_1 < 0{\cdot}45$, $1{\cdot}71 < \alpha_3 < 1{\cdot}84$, whence $6{\cdot}39 < a < 6{\cdot}94$, which is impossible.

Type (0, 0, 2, 4). In this case $S > 30$ and, since $a \not\equiv 2 \pmod 4$, $S \not\equiv 4 \pmod 8$. If $a = 8$, $S = 32$, then $0{\cdot}66 < \alpha_1$ and $\alpha_3 < 2{\cdot}68$ which contradicts I_1.

If $S = 35$ then $\alpha_4 < 4{\cdot}27$ and, by I_1, $\alpha_1 < 0{\cdot}61$ whence $a \neq 9$. Hence $a = 7$ and we have $\alpha_1 > 0{\cdot}42$, $\alpha_3 < 2{\cdot}16$ which contradicts I_2.

Type (0, 0, 3, 4). By I_1, $\alpha_1 < 0{\cdot}72$ whence, if $S \leqslant 36$, we have $\alpha_4 < 4{\cdot}2$. Hence $a \leqslant 9$ and $S > 32$ whence $a \neq 8$. But if $a = 9$ and $\alpha_1 < 0{\cdot}72$ then $S > 37$.

Type (0, 1, 1, 4). By I_1, I_2 we have $1{\cdot}50 < \alpha_2 < 1{\cdot}60$, whence $\alpha_1 < 0{\cdot}31$ and $a = 8$, so that we need consider only $b = 20$. Also $f(1) = f(2) = -1$ so that $f(x) = x^4 - 8x^3 + 20x^2 - 19x + 5$. But this has a negative discriminant.

Type (0, 2, 2, 4). By I_2, I_3, $2{\cdot}40 < \alpha_2 < 2{\cdot}60$. If $a = 9$, then $\alpha_4 < 4{\cdot}20$, whence $\Pi\,|(\alpha_i-2)(\alpha_i-3)| < (\frac{1}{4})^2\,(6)(2{\cdot}64) < 1$, which is false.

If $a = 10$, then we may suppose by symmetry that $\alpha_4 - 4 \leqslant 1 - \alpha_1$ whence $S \leqslant 36$, $\alpha_4 < 4{\cdot}63$ and from this $f(2) = f(3) = -1$, $f(4) \geqslant -6$. Trial of the various possibilities for $f(4)$ shows that only $f(4) = -5$ gives all α's real and that $f(x)$ is then

$$x^4 - 10x^3 + 33x^2 - 40x + 11 = (x^2 - 3x + 1)(x^2 - 7x + 11).$$

Types (0, 1, 2, 5) *and* (0, 1, 3, 5). In these cases

$$-8 \geqslant 3f(1) - 4f(2) + f(5) = 564 - 96a + 12b,$$

whence $S \geqslant 43$ for $9 \leqslant a \leqslant 11$.

Type (0, 2, 3, 5). In this case,

$$-4 \geqslant f(1) - 2f(3) + f(5) = 464 - 72a + 8b$$

whence $S \geqslant 40$ for $11 \leqslant a \leqslant 13$.

Type (0, 2, 2, 5). If $S \leqslant 36$ then $\alpha_4 < 5{\cdot}24$, $0{\cdot}75 < \alpha_1$. Hence, by I_2, I_3, $2{\cdot}49 < \alpha_2 < 2{\cdot}56$. Hence $a = 11$, $f(1) = -1$ or -2, $f(2) = f(3) = -1$. Since $f(1) \equiv f(3)$ modulo 2, we have $f(1) = -1$ and

$$f(x) = x^4 - 11x^3 + 41x^2 - 61x + 29.$$

But this has a negative discriminant.

Types with $[\alpha_4] < 3$. Since $|x(x-1)(x-2)(x-3)| < 1$ in (0, 3) except for $x = \frac{1}{2}(3 \pm \sqrt{5})$ we can only, by I_0, I_1, I_2, I_3, have $\alpha_1 = \alpha_2 = \frac{1}{2}(3 - \sqrt{5})$, $\alpha_3 = \alpha_4 = \frac{1}{2}(3 + \sqrt{5})$ and then $f(x) = f_1(x) = (x^2 - 3x + 1)^2$.

Type (0, 0, 2, 3). I_0, I_1, I_2, I_3, I_4 give

$$0{\cdot}28 < \alpha_1 < 0{\cdot}60, \quad 2{\cdot}16 < \alpha_3 < 2{\cdot}87, \quad 3{\cdot}16 < \alpha_4 < 3{\cdot}97.$$

If $a = 6$ then $\alpha_4 < 3{\cdot}28$ whence $\alpha_3 < 2{\cdot}52$, $0{\cdot}34 < \alpha_1$ and $a > 6$, which gives a contradiction. If $0{\cdot}57 \leqslant \alpha_1$ then $\alpha_3 < 2{\cdot}83$ and $a \leqslant 7$. If $\alpha_1 < 0{\cdot}57$ then $a \leqslant 7$. Hence $a = 7$. Also $f(0) \leqslant 4$, $f(1) \leqslant 2$, $f(2) \leqslant 5$. If $f(1) = 2$ then $\alpha_1 < 0{\cdot}40$ by I_1 and $f(0) = 1$. Now

$$f(x) = x^4 - 7x^3 + 14x^2 - 8x + \tfrac{1}{2}(x-1)(x-2)f(0) - x(x-2)f(1) + \tfrac{1}{2}x(x-1)f(2)$$

and we need $$b = 14 + \tfrac{1}{2}\big(f(0) + f(2)\big) - f(1) \geqslant 14,$$

and $$f(3) = -6 + f(0) - 3f(1) + 3f(2) \leqslant -1.$$

Possibilities for $f(0)$, $f(1)$, $f(2)$ [remembering that $f(0) \equiv f(2)$ modulo 2] are (3, 1, 1), (2, 1, 2), (1, 1, 1) and (1, 2, 3).

These give respectively

$$f(x) = x^4 - 7x^3 + 15x^2 - 11x + 3, \text{ with negative discriminant;}$$

$$f(x) = x^4 - 7x^3 + 15x^2 - 10x + 2 = (x^2 - 3x + 1)(x^2 - 4x + 2), \text{ reducible;}$$

$$f(x) = f_2(x) = x^4 - 7x^2 + 14x - 8x + 1;$$

$$f(x) = x^4 - 7x^3 + 14x^2 - 7x + 1 = (x^2 - 3x + 1)(x^2 - 4x + 1), \text{ reducible.}$$

Type (0, 0, 3, 3). By I_0, I_1, $0{\cdot}25 < \alpha_1 < 0{\cdot}67$; similarly

$$3{\cdot}33 < \alpha_3 < 3{\cdot}75,$$

whence $a = 8$ and so $S = 32$. We may suppose by symmetry that $\alpha_3 - 3 \leqslant 1 - \alpha_1$ so that $\alpha_3 < 3{\cdot}42$, whence $0{\cdot}29 < \alpha_1$, by I_0. Hence $f(3) = 1$, $f(1) \leqslant 2$ and since $f(1) \equiv f(3)$ (mod 2),

$$f(x) = f_3(x) = x^4 - 8x^3 + 20x^2 - 16x + 4 = (x^2 - 4x + 2)^2.$$

Type (0, 1, 2, 3). We have

$$44 \leqslant 36 + f(0) - 3f(1) + 3f(2) - f(3) = 6a = 60 + f(1) - 3f(2) + 3f(3) - f(4)$$

$$\leqslant 52, \text{ whence } a = 8.$$

Then

$$236 \leqslant 3f(0) - 4f(1) + f(4) + 228 = 12b = 276 + 6f(1) - 12f(2) + 6f(3)$$

$$\leqslant 252, \text{ whence } 20 \leqslant b \leqslant 21.$$

If $b = 20$ then $47 \leqslant f(0) - f(3) + 45 = 3c = 51 + f(1) - f(4) \leqslant 49$, whence $c = 16$. Finally $1 \leqslant f(0) = d = f(1) + 3 \leqslant 2$. Thus we obtain

$$f(x) = f_4(x) = x^4 - 8x^3 + 20x^2 - 16x + 1,$$

and

$$f(x) = f_5(x) = x^4 - 8x^3 + 20x^2 - 16x + 2.$$

If $b = 21$ then $20 \leqslant 18 + f(2) - f(3) = c = 22 + f(1) - f(2) \leqslant 20$. Finally

$$5 \leqslant f(2) + 4 = d = f(1) + 6 \leqslant 5.$$

This gives $x^4 - 8x^3 + 21x^2 - 20x + 5 = (x^2 - 3x + 1)(x^2 - 5x + 5)$.

Type (0, 1, 2, 4). By reasoning similar to that in Type (0, 1, 2, 3) we obtain bounds for a, b, c, d successively and find that if $S \leqslant 36$, $f(x)$

can only be

$$f_6(x) = x^4-8x^3+20x^2-17x+3; \quad f_7(x) = x^4-9x^3+26x^2-26x+5;$$

$$x^4-9x^3+26x^2-27x+7 = (x^2-3x+1)(x^2-6x+7);$$

$$f_8(x) = x^4-9x^3+26x^2-27x+8; \quad f_9(x) = x^4-9x^3+26x^2-28x+9;$$

$$f_{10}(x) = x^4-9x^3+27x^2-31x+11; \quad \text{or} \quad f_{11}(x) = x^4-10x^3+33x^2-42x+17.$$

Type (0, 1, 3, 4). Here we find that $f(x)$ can only be

$$f_{12}(x) = x^4-9x^3+26x^2-26x+7;$$

$$f_{13}(x) = x^4-10x^3+33x^2-39x+11; \quad f_{14}(x) = x^4-10x^3+33x^2-40x+13;$$

$$x^4-10x^3+33x^2-40x+14 = (x^2-4x+2)(x^2-6x+7);$$

$$x^4-10^3+33x^2-40x+15 = (x^2-5x+3)(x^2-5x+5);$$

or

$$f_{15}(x) = x^4-10x^3+34x^2-45x+19.$$

We have now to examine $f_1(x), \ldots, f_{15}(x)$ in detail. $f_1(x)$ shows that some fields with discriminant less than 2000 may have $k(\sqrt{5})$ as a subfield: $f_2(x)$, $f_{10}(x)$, $f_{15}(x)$ are reducible in $k(\sqrt{5})$ and so arise from such fields. $f_3(x)$ shows that fields with the subfield $k(\sqrt{2})$ may arise and $f_5(x)$, $f_{11}(x)$ are reducible in $k(\sqrt{2})$. $f_4(x)$ and $f_{14}(x)$ are reducible in $k(\sqrt{3})$.

The polynomials $f_6(x)$, $f_9(x)$ and $f_{12}(x)$ have discriminant 1957; since 1957 is squarefree the zeros of these polynomials generate fields with discriminant 1957: the fields are in fact conjugates of each other since the substitution $x = (2y-3)/(y-2)$ in $f_9(x) = 0$ gives $f_6(y) = 0$, while the substitution $x = (y-2)^2$ in $f_9(x) = 0$ gives $f_{12}(y) f_{12}(4-y) = 0$.

The polynomials $f_8(x)$ and $f_{13}(x)$ have discriminant 2777, while $f_7(x)$ has discriminant 3981: similarly there is just one field with its conjugates for each of these discriminants.

Now if μ is squarefree and belongs to $k(\sqrt{5})$ then the discriminant of the quartic field $K(\sqrt{\mu})$ is $25n(\mu)$, if $\mu \equiv \nu^2$ modulo 4 is solvable for an integer ν of $k(\sqrt{5})$, and $400n(\mu)$ otherwise, where $n(\mu)$ is the norm of μ. By trial we find that the discriminant of $K\left(\sqrt{(7+2\sqrt{5})}\right)$ is 725, of $K\left(\sqrt{(15+6\sqrt{5})}\right)$ is 1125 and of $K\left(\sqrt{(3+\sqrt{5})}\right)$ is 1600 (note that $\sqrt{(3+\sqrt{5})} = \frac{1}{2}(1+\sqrt{5})\sqrt{2}$, so that $k(\sqrt{2})$ is also a subfield of $K\left(\sqrt{(3+\sqrt{5})}\right)$). No other field with discriminant less than 2000 has $k(\sqrt{2})$ as a subfield and no field with $k(\sqrt{3})$ as a subfield has discriminant less than 2000. This completes the proof of Theorem 2.

More extensive calculations, considering values of $S \leqslant 68$ (*i.e.* $S < 72$) give all discriminants less than 11664. The only point of procedure not exemplified in the proof of Theorem 2 is the elimination of fields with discriminants greater than 11664 when these are not squarefree. This can be done by testing for integral bases other than 1, θ, θ^2, θ^3 [where θ is a zero of the polynomial $f(x)$ which has been discovered] or in some cases by factorizing rational primes in the field or by noting that the defining polynomial is an Eisenstein polynomial for some prime (see [3], pp. 154, 322 and 460).

Table I below gives the fields of type $K(\sqrt{\mu})$ where μ is a quadratic integer, Table II defining polynomials for the fields with no subfield. In every case in Table II an integral basis is provided by 1, θ, θ^2, θ^3, where θ generates the field.

No case arises for discriminants less than 11664 of two distinct (*i.e.* non-conjugate) fields with the same discriminant: however 16448 is the discriminant of $K\left(\sqrt{(17+4\sqrt{2})}\right)$ and of the quartic field with no subfield generated by zeros of $x^4-10x^3+30x^2-32x+10$. It may well be that 16448 is the least discriminant for which there are two distinct fields.

TABLE I.

Δ	μ	Δ	μ	Δ	μ
725	$7+2\sqrt{5}$	4752	$9+4\sqrt{3}$	8525	$19+2\sqrt{5}$
1125	$15+6\sqrt{5}$	4913	$17+4\sqrt{17}$	8725	$23+6\sqrt{5}$
1600	$3+\sqrt{5}$	5125	$15+2\sqrt{5}$	8768	$13+4\sqrt{2}$
2000	$5+2\sqrt{5}$	5225	$17+4\sqrt{5}$	8957	$51+14\sqrt{13}$
2048	$2+\sqrt{2}$	5725	$27+10\sqrt{5}$	9225	$33+12\sqrt{5}$
2225	$13+4\sqrt{5}$	6125	$35+14\sqrt{5}$	9248	$\frac{1}{2}(5+\sqrt{17})$
2304	$2+\sqrt{3}$	7056	$49+28\sqrt{3}$	9792	$15+6\sqrt{2}$
2525	$11+2\sqrt{5}$	7168	$3+\sqrt{2}$	10025	$41+16\sqrt{5}$
2624	$7+2\sqrt{2}$	7225	$153+68\sqrt{5}$	10304	$17+8\sqrt{2}$
3600	$27+12\sqrt{5}$	7232	$11+2\sqrt{2}$	10309	$23+6\sqrt{13}$
4205	$11+2\sqrt{29}$	7488	$5+2\sqrt{3}$	10512	$29+16\sqrt{3}$
4225	$117+52\sqrt{5}$	7600	$8+3\sqrt{5}$	10816	$39+26\sqrt{2}$
4352	$5+2\sqrt{2}$	7625	$25+8\sqrt{5}$	11025	$189+84\sqrt{5}$
4400	$4+\sqrt{5}$	8000	$5+\sqrt{5}$	11525	$31+10\sqrt{5}$
4525	$19+6\sqrt{5}$	8112	$4+\sqrt{13}$	11661	$11+2\sqrt{13}$

TABLE II.

	a	b	c	d		a	b	c	d
1957	8	20	17	3	8789	9	24	22	5
2777	8	19	13	2	9301	9	25	23	5
3981	9	25	21	3	9909	9	24	18	3
5744	8	19	14	1	10273	9	22	13	2
6224	8	18	10	1	10889	9	25	22	1
6809	9	24	17	2	11197	9	23	13	1
7053	10	32	33	3	11324	11	40	65	23
7537	8	17	9	1	11344	10	30	26	6
8069	9	22	10	1	11348	9	25	23	4
8468	9	25	21	2					

References.

1. B. N. Delone and D. K. Faddeev, *Acad. Sci. U.R.S.S. Trav. Inst. Math. Stekloff*, 11 (1940).
2. H. J. Godwin, *Journal London Math. Soc.*, 25 (1950), 331–339.
3. H. Hasse, *Zahlentheorie* (Berlin, 1949).
4. J. Mayer, *Sitzungsberichte Wien*, 138 (1929), 733–742.
5. J. Sommer, *Introduction a la théorie des nombres algébriques* (Paris, 1911).

University College of Swansea.

ON THE INNER PRODUCT OF S-FUNCTIONS

A. G. KAKAR*.

The direct product of two representations of a symmetric group is equivalent to a direct sum of irreducible representations. This may be expressed as

$$[\lambda]\times[\mu]=\Sigma g_{\lambda\mu\nu}[\nu],$$

where (λ), (μ), (ν) denote partitions of n, and $[\lambda]$, $[\mu]$, $[\nu]$ denote the corresponding matrix representations. Equivalently

$$\chi^{(\lambda)}\chi^{(\mu)}=\Sigma g_{\lambda\mu\nu}\chi^{(\nu)}.$$

Littlewood (1) has defined the inner product of S-functions as

$$\{\lambda\}\circ\{\mu\}=\Sigma g_{\lambda\mu\nu}\{\nu\}$$

which is another way of expressing the same result.

Murnaghan (2) has tabulated results which correspond to the expansions of

$$\{n-r, \lambda\}\circ\{n-s, \mu\}$$

where (λ), (μ) denote partitions of r and s respectively for all cases for which $r=1, 2, 3, 4$ and $s=1, 2, 3$.

Murnaghan's methods are rather tentative. Littlewood has given a more explicit method of calculating $\{\lambda\}\circ\{\mu\}$ by an S-functional analysis.

* Received 23 September, 1955; read 17 November, 1955.