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so that \( g(x) \) is an integral function of \( x \); write also

\[
I_n(x) = (1 + x)^{-n-1} = \sum_{m=0}^{\infty} (-1)^m \binom{n + m}{m} x^m, \quad (m = 0, 1, 2, \ldots),
\]

\[
f(x) = \sum_{m=0}^{\infty} \binom{k}{\phi(m)} z^m f_m(x).
\]

It is easily shown \( f(x) \) is analytic for \( |x| < 1 \), and that, as \( x \) tends along any non-constant path to a point \( x_0 \) on the unit-circle, Abel's limit

\[
\lim_{x \to x_0} f(x) = \frac{1}{1 - x_0} g \left( \frac{1}{1 + x_0} \right), \quad (|x_0| = 1; \ x_0 \neq -1)
\]

exists. On the other hand we have, for \( |x| < 1 \),

\[
f(x) = \sum_{m=0}^{\infty} \sigma_m x^m,
\]

where

\[
(-1)^m \sigma_m = \sum_{m=0}^{\infty} \binom{k}{\phi(m)} \binom{n + m}{m},
\]

and therefore

\[
(2 \text{ bis}) \quad |\sigma_m| > \sum_{m=0}^{\infty} \binom{k}{\phi(m)} z^m > e^{n \phi(m)},
\]

in virtue of \( \S \).


\[
\text{SOME PROBLEMS IN POTENTIAL THEORY.}
\]

By Dr. H. Bateman.

\( \S \). In a previous note\( \dagger \) it was shown that the potential of a surface of revolution, whose meridian curve is a limaçon, can be expressed in the form

\[
V = (\cosh \sigma - \cos \chi) \sum_{n=0}^{\infty} \frac{P_n(\cosh \sigma)}{\sinh(n+1) \cosh \sigma} Q_n(\cos \chi) P_n(\cos \chi),
\]

the potential being unity over the surface \( \sigma = \sigma_1 \), where

\[
\left( \frac{R + X}{2} \right) = \frac{a \sinh \sigma}{\cosh \sigma - \cos \chi} \quad \left( \frac{R - X}{2} \right) = \frac{a \sin \chi}{\cosh \sigma - \cos \chi}
\]

To find the capacity of the surface we must determine the form of \( 1 \) at infinity, i.e. in the neighbourhood of \( \sigma = 0 \), \( \chi = 0 \). Writing

\[
P_n(\cosh \sigma) = \frac{1}{2} \sum_{m=0}^n \frac{(n+1)(n+2)}{2} (\cosh \sigma - 1)^{m+1} \quad \frac{(2n+1)}{2} (\cosh \sigma - 1)^{m+1}
\]

\[
P_n(\cos \chi) = 1 + \frac{n+1}{2} (\cos \chi - 1)^{m+1} \quad \frac{(n+1)(n+2)}{2} (\cos \chi - 1)^{m+1}
\]

\[
R = a^2 \frac{\cosh \sigma + \cos \chi}{\cosh \sigma - \cos \chi} \sim a^2 \frac{\cosh \sigma + \cos \chi}{\cosh \sigma - \cos \chi}^3
\]

\[
X = a^2 \frac{\cosh \sigma + \cos \chi - 2}{\cosh \sigma - \cos \chi} \sim 2a^2 \frac{\cosh \sigma + \cos \chi - 2}{\cosh \sigma - \cos \chi}^3
\]

we find that

\[
V = \frac{2a^2}{R} \sum_{n=0}^{\infty} \frac{Q_n(\cosh \sigma)}{P_n(\cosh \sigma)} = \frac{2a^2}{R} \sum_{n=0}^{\infty} \frac{(2n+1)}{2} \frac{Q_n(\cosh \sigma)}{P_n(\cosh \sigma)}
\]

and \( \cdots \).

The first term gives an expression for the capacity \( C \), viz.,

\[
C = a^2 \sum_{n=0}^{\infty} \frac{Q_n(\cosh \sigma)}{P_n(\cosh \sigma)}
\]

while the second term enables us to determine a point where the charge \( C \) should be placed in order that its potential may agree with \( V \) at infinity up to terms of the second order in \( \frac{1}{R} \).

This point may be called the centre of charge.

To find the polar equation of the limaçon we write

\[
r = \frac{2a^2}{\cosh \sigma + \cos \chi}, \quad \cos \theta = \frac{\cosh \sigma \cos \chi - 1}{\cosh \sigma - \cos \chi}, \quad \sin \theta = \frac{\sinh \sigma \sin \chi}{\cosh \sigma - \cos \chi},
\]

then

\[
X = a^2 + r \cos \theta, \quad Y = \sqrt{(R^2 - X^2)} = r \sin \theta,
\]

and

\[
r = 2a^2 \frac{\sinh \sigma}{\sinh \sigma + \cos \theta}.
\]

The area of the surface generated by the revolution of the limaçon about its axis of symmetry is \( 4\pi k^2 \), where

\[
k = 2a^2 \cosh \sigma \left( \cosh \sigma + \frac{\pi}{3} \right).
\]

With the aid of tables for \( Q_n(\cosh \sigma) \) and \( P_n(\cosh \sigma) \) we find that

\[
\begin{array}{ccc}
\sigma_1 & C/2a^2 & k/2a^2 \\
2 & .718605 & .722009 \\
1.2 & 3.25824 & 3.29872
\end{array}
\]

In the case of a sphere \( \cosh \sigma = \infty \) we have, of course, \( C = k \).

Of all surfaces of given area the sphere has apparently the greatest capacity. When \( \cosh \sigma = 2 \) the limaçon has a point of undulation on the axis of symmetry, the points of contact of the double tangent being consecutive. The value of \( C \) in this case differs from \( k \) by about 1 part in 200. When \( \cosh \sigma = 1.2 \) the double tangent touches the limaçon in two distinct real points, and the curve bends inwards near the vertex. The capacity is slightly reduced by this hollow, \( C \) differing from \( k \) by about 1 part in 80.

\[\text{\textsection 2.}\]

Since the author does not remember having seen any tables of spheroidal harmonics, the values of \( P_n, Q_n \) and their first derivatives are given\(^*\) for a few values of \( \cosh \sigma \).

\[
\begin{array}{ccc}
n & P_n(\sigma) & Q_n(\sigma) \\
0 & 1 & 1.52226 12188 \\
1 & 1.1 & 6.7448 73407 \\
2 & 1.315 & 3.5177 35028 \\
3 & 1.6775 & .19525 98613 \\
4 & 2.24283 75 & .11204 51059 \\
5 & 3.09901 625 & .05654 14207 \\
6 & 4.38056 81875 & .03900 59434 \\
7 & 6.29527 53687 & .02341 94953 \\
8 & 9.14543 59340 & .01417 25085 \\
9 & 12.40879 97039 & .00862 99941 \\
10 & 17.28347 69907 & .00528 14300 \\
11 & 29.37649 19495 & .00324 55538 \\
12 & 43.79141 66188 & .00200 13984 \\
13 & 65.51892 72018 & .00123 78316 \\
14 & 96.33026 58463 & .00076 75299 \\
15 & 147.94469 99781 & .00047 69708 \\
16 & 223.16514 25975 & .00029 69847 \\
17 & 337.26282 21552 & .00018 52360 \\
18 & 510.59955 43788 & .00011 57137 \\
19 & 774.24631 91802 & .00007 23842 \\
20 & 1175.68877 79816 & .00004 53361
\end{array}
\]

* In calculating these values use has been made of the values of \( \log 2, \log 3, \log 5, \log 7, \) and \( \log 10 \), given by J. C. Adams, Proc. Roy. Soc. London, vol. xxvii. (1678), p. 68.
Since these values were calculated with the aid of the

\[ P_{n} - P_{n+1} = (2n + 1) P_{n} \]

\[ Q_{n} - Q_{n+1} = (2n + 1) Q_{n} \]

the last two or three figures in the above numbers are doubtful when \( n \) is large. The difference relations

\[ P_{n} - s P_{n+1} = n P_{n} \]

\[ Q_{n} - s Q_{n+1} = n Q_{n} \]

are, however, satisfied to 9 decimal places when \( n = 20 \), so the last figure may be the only one which is wrong.

\[ s = \cosh \sigma = 1.1 \]

\[ \begin{array}{c|c|c|c}
 n & P_{n}(s) & P_{n}(s) & Q_{n}(s) \\
 \hline
 0 & 0 & 1.19894 & 76364 \bar{9} \\
 1 & 1.2 & 1.23737 & 15132 \bar{9} \\
 2 & 2.65 & 2.9573 & 89 \bar{3} \\
 3 & 3.52 & 4.76 & 132 \bar{9} \\
 4 & 4.0 & 6.952 & 70 \bar{8} \\
 5 & 6.25 & 11.925 & 82 \bar{9} \\
 6 & 11.43 & 18.125 & 82 \bar{9} \\
 7 & 19.66 & 30.725 & 82 \bar{9} \\
 8 & 34.31 & 55.398 & 82 \bar{9} \\
 9 & 60.27 & 92.749 & 82 \bar{9} \\
 10 & 106.54 & 142.258 & 82 \bar{9} \\
\end{array} \]

\( \frac{\partial P}{\partial N} = U \frac{\partial X}{\partial N} \)

over the surface \( \sigma = \sigma \), we assume for points outside the body

\[ V = \alpha' U (cosh \sigma - cos \chi) \sum_{m=0}^{\infty} (2m+1) A_m P_m(cosh \sigma) P_m(cos \chi) \]

\[ = \alpha' U \sum_{m=0}^{\infty} (m+1) (A_m - A_{m-1}) \]

Now

\[ X = \alpha' \sinh \sigma - \sin \chi \]

\( (cosh \sigma - cos \chi)^{1/2} \)

\[ = \alpha' (cosh \sigma - cos \chi) \sum_{m=0}^{\infty} (2m+1) \{ Q_m(cosh \sigma) P_{m+1}(cos \chi) - Q_{m-1}(cosh \sigma) P_m(cos \chi) \} \]

\[ = \alpha'^2 \sum_{m=0}^{\infty} (m+1)! \{ Q_m(cosh \sigma) P_{m+1}(cos \chi) - Q_{m-1}(cosh \sigma) P_m(cos \chi) \} \]
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hence the boundary condition at \( \sigma = \sigma_0 \) will be satisfied for all values of \( \chi \) if

\[
U \sum_{m=0}^{\infty} (m+1) \left( A_{m} - A_{-m} \right) \left( P_{m} \cosh \sigma_0 \right) \frac{Q_{m} \cosh \sigma_0}{P_{m} \left( \cosh \sigma \right)} - \frac{P_{m} \left( \cosh \sigma \right) P_{m} \left( \cosh \sigma_0 \right)}{P_{m} \left( \cosh \sigma_0 \right) P_{m} \left( \cosh \sigma \right)}
\]

This leads to the system of equations

\[
m \left( A_{m} - A_{-m} \right) \left( P_{m} \cosh \sigma_0 \right) - (m+1) \left( A_{m} - A_{-m} \right) \left( P_{m} \cosh \sigma_0 \right) = 2mnQ_{m} \cosh \sigma_0 - 2 \left( m+1 \right) Q_{m} \cosh \sigma_0.
\]

The left-hand side of the typical equation becomes a perfect difference when multiplied by \( P_{m} \cosh \sigma_0 \), while the right-hand side may be transformed with the aid of the identity

\[
Q_{m} \cosh \sigma_0 \left( \frac{P_{m} \cosh \sigma_0}{P_{m} \cosh \sigma_0} - \frac{Q_{m} \cosh \sigma_0}{P_{m} \cosh \sigma_0} \right) = m \cosh \sigma_0.
\]

Consequently the typical equation may be written in the form

\[
m \left( A_{m} - A_{-m} \right) \left( P_{m} \cosh \sigma_0 \right) \left( P_{m} \cosh \sigma_0 \right) - (m+1) \left( A_{m} - A_{-m} \right) \left( P_{m} \cosh \sigma_0 \right)^2 = 2mnQ_{m} \cosh \sigma_0 P_{m} \left( \cosh \sigma_0 \right).
\]

Summing from \( m = 1 \) to \( m = n \), we get

\[
\left( m+1 \right) \left( A_{m} - A_{-m} \right) \left( P_{m} \cosh \sigma_0 \right) \left( P_{m} \cosh \sigma_0 \right) = 2 \left( m+1 \right)^2 Q_{m} \cosh \sigma_0 P_{m} \left( \cosh \sigma_0 \right) + \frac{n \left( n+1 \right)}{2} \sinh^2 \sigma_0,
\]

therefore

\[
A_{m} - A_{-m} = 2 \left( m+1 \right) Q_{m} \cosh \sigma_0 P_{m} \left( \cosh \sigma_0 \right) + \frac{n \left( n+1 \right)}{2} \sinh^2 \sigma_0.
\]

Hence finally we obtain the following expression for \( V \)

\[
V = \frac{1}{2} a^2 U \sum_{m=0}^{\infty} \left( m+1 \right)^2 \left( P_{m} \cosh \sigma_0 \right) \left( \frac{Q_{m} \cosh \sigma_0}{P_{m} \left( \cosh \sigma_0 \right)} - \frac{P_{m} \cosh \sigma_0}{P_{m} \left( \cosh \sigma_0 \right)} \right)
\]

**We may deduce from this expression the form which \( \Phi \) takes at infinity by writing for small values of \( \sigma \) and \( \chi \) the expansions for \( P_{m} \left( \cosh \sigma_0 \right) \) and \( P_{m} \left( \cosh \chi \right) \) used before. The coefficient of \( \cosh \sigma \cos \chi \) is then**

\[
a^2 U \sum_{m=0}^{\infty} \left( m+1 \right)^2 \left( P_{m} \cosh \sigma_0 \right) \left( \frac{Q_{m} \cosh \sigma_0}{P_{m} \left( \cosh \sigma_0 \right)} - \frac{P_{m} \cosh \sigma_0}{P_{m} \left( \cosh \sigma_0 \right)} \right)
\]

and this is zero. The most important term in the expansion is thus

\[
a^2 U \left( \cosh \sigma - \cos \chi \right) \left( \cosh \sigma + \cos \chi - 2 \right) \sum_{m=0}^{\infty} \left( m+1 \right)^2 \left( m+2 \right) \left( \frac{Q_{m} \cosh \sigma_0}{P_{m} \left( \cosh \sigma_0 \right)} - \frac{P_{m} \cosh \sigma_0}{P_{m} \left( \cosh \sigma_0 \right)} \right)
\]

**Now**

\[
X = \frac{1}{a^2} \cosh \sigma + \cos \chi - 2 \left( \cosh \sigma + \cos \chi \right)^2,
\]

\[
J_0 = a^2 \left( \cosh \sigma - \cos \chi \right)^2.
\]

\[
= \frac{1}{a^2} \left( \cosh \sigma + \cos \chi - 2 \left( \cosh \sigma - \cos \chi \right) \right) \left( \cosh \sigma + \cos \chi - 2 \left( \cosh \sigma - \cos \chi \right) \right)
\]

\[
= \frac{1}{4a^2} \left( \cosh \sigma - \cos \chi \right) \left( \cosh \sigma + \cos \chi - 2 \right)
\]

hence the most important part of the expansion is equal to

\[
\frac{1}{2} a^2 U \sum_{m=0}^{\infty} \left( m+1 \right)^2 \left( m+2 \right) \left( \frac{Q_{m} \cosh \sigma_0}{P_{m} \left( \cosh \sigma_0 \right)} - \frac{P_{m} \cosh \sigma_0}{P_{m} \left( \cosh \sigma_0 \right)} \right)
\]

**This gives the moment of the doublet whose potential is a first approximation to the value of \( V \) at infinity. The apparent mass of the fluid may be found by means of a theorem due to**

\[
\rho B \left( 2 \pi a^2 \sum_{m=0}^{\infty} \left( m+1 \right)^2 \left( m+2 \right) \left( \frac{Q_{m} \cosh \sigma_0}{P_{m} \left( \cosh \sigma_0 \right)} - 1 \right) \right),
\]

where \( B \) is the volume of the fluid displaced by the solid, \( \rho \) the density of the fluid. Since
\[
B = \frac{4\pi}{3} \cdot 8a^2 \cosh^2 \sigma_0 \left[ 1 + \frac{1}{\cosh^2 \sigma_0} \right],
\]
we find that the apparent mass is \( k \rho B \), where \( k = .5 \) for the sphere. When
\[
cosh \sigma_0 = 1.2 \quad \text{we find} \quad k = .5688, \\
cosh \sigma_0 = 2 \quad \text{thus} \quad k = .548, \\
cosh \sigma_0 = 3 \quad \text{thus} \quad k = .527.
\]

A GENERAL FORM OF THE REMAINDER IN TAYLOR'S THEOREM.

By G. S. Mahajani, St. John's College, Cambridge.

1. An examination of the various extant accounts of Taylor's theorem reveals that, for the most part, they obtain the particular form of the remainder with which they happen to be concerned by utilising what we may call the simple form of the mean value theorem, which states that if \( f(x) \) is continuous in the interval \((a, b)\), end points included, and differentiable in the same interval, end points not necessarily included, then
\[
f(b) - f(a) = (b - a)/\xi \left( f'(\xi) \right),
\]
where \( \xi \) is some number between \( a \) and \( b \) and not coinciding with either. Now it is well known that the mean value theorem can be expressed in a form more general than the above. If \( \phi(x) \) satisfies the same conditions as \( f(x) \) and, in addition, is such that \( \phi'(x) \) does not vanish anywhere in \((a, b)\), then
\[
f(b) - f(a) = (b - a) \phi'(\xi),
\]
where \( \xi \), not necessarily the same as before, lies between \( a \) and \( b \) and does not coincide with either of them.

We propose to show that, by utilising this more general form of the mean value theorem, we can obtain an extremely general form of the remainder in Taylor's theorem.

2. We suppose that \( f(x) \) satisfies the strict conditions of order \( n+1 \) at \( a \), being such that it and its first \( n+1 \) derivatives exist in some neighbourhood of \( a \); and that \( \phi(x) \) satisfies
the conditions of order \( p+1 \) at \( a \). Further, we suppose that \( \phi^{(n+1)}(x) \) does not vanish.

3. Let
\[
f(a + h) = f(a) + hf'(a) + \ldots + \frac{h^n}{n!} f^{(n)}(a) + R_n(a),
\]
so that \( R_n \) is the usual remainder. Evidently
\[
R_n = f(a + h) - f(a) - hf'(a) - \ldots - \frac{h^n}{n!} f^{(n)}(a) \ldots (1).
\]

4. Write now
\[
\psi(x) = f(a + h) - f(x) - (a + h - x)f'(x) - \ldots - \frac{(a + h - x)^n}{n!} f^{(n)}(x)
\]
\[
\ldots \ldots (2),
\]
\[
\chi(x) = \phi(a + h) - \phi(x) - (a + h - x)\phi'(x) - \ldots - \frac{(a + h - x)^p}{p!} \phi^{(p)}(x)
\]
\[
\ldots \ldots (3).
\]

Then, as is easily seen,
\[
\psi(x) = \frac{(a + h - x)^n}{n!} \phi^{(n+1)}(x),
\]
\[
\chi(x) = \frac{(a + h - x)^p}{p!} \phi^{(p+1)}(x).
\]

5. By the mean value theorem in its general form,
\[
\psi(a + h) - \psi(a) = \frac{\psi'(\xi)}{\chi'(\xi)}
\]
\[
\chi'(\xi) = \frac{(a + h - x)^p}{p!} \phi^{(p+1)}(x),
\]
where \( \xi \) lies between \( a \) and \( a + h \) and coincides with neither. In the usual way we have
\[
\xi = a + \theta h,
\]
where
\[
0 < \theta < 1.
\]

Further, as is easily seen,
\[
\psi'(a + h) = \chi(a + h) = 0.
\]

Thus
\[
\psi(a) = \psi(a + \theta h) = \frac{\psi'(a + \theta h)}{\chi'(a + \theta h)} = \frac{p!}{n!} (h - \theta h)^{n} \phi^{(n+1)}(a + \theta h) \phi^{(p+1)}(a + \theta h),
\]
\[
\chi(a) = \chi(a + \theta h) = \frac{\chi'(a + \theta h)}{\phi'(a + \theta h)} = \frac{p!}{n!} (h - \theta h)^{n} \phi^{(n+1)}(a + \theta h) \phi^{(p+1)}(a + \theta h).
\]

6. But (1) and (2) give at once \( \psi(a) = R_n(a) \). Thus
\[
R_n = \frac{p!}{n!} (h - \theta h)^{n} \phi^{(n+1)}(a + \theta h) \chi(a) \phi^{(p+1)}(a + \theta h),
\]
\[
= \frac{p!}{n!} (h - \theta h)^{n} \phi^{(n+1)}(a + \theta h) \left\{ \phi(a + \theta h) - \phi(a) - \ldots - \frac{p!}{p!} \phi^{(p)}(a) \right\}.
\]