2569

JIMS 6 (1942)

ON THE MAXIMUM VALUE OF THE NUMBER OF PARTITIONS OF n INTO k PARTS.

BY

F. C. AULUCK, S. CHOWLA, and H. GUPTA.

[Received 26 January 1942.]

1. Let $p_k(n)$ denote the number of partitions of n into *exactly* k parts. We obviously have

$$\sum_{k=1}^{n} p_k(n) = p(n),$$

where $p(n)$ denotes the number of unrestricted partitions of n, a function introduced by Euler. Hardy and Ramanujan proved the asymptotic formula

$$p(n) \sim \frac{e^{c\sqrt{n}}}{(4\sqrt{3})n}, \text{ where } c = \pi\sqrt{\tfrac{2}{3}}.$$

They also proved a formula from which $p(n)$ can be estimated with great rapidity for fairly large values of n. D. H. Lehmer has used the Hardy-Ramanujan formula to calculate $p(14031)$. Recently Rademacher* has found an 'exact' formula for $p(n)$, suggested by the formula of Hardy and Ramanujan.

Some months ago, at the suggestion of Dr. Kothari, we undertook the study of $p_k(n)$. Although our first results were clumsy, our investigations got a fillip from the recent paper of Erdos and Lehner†, which contains the remarkable result that, denoting $\sum_{r=1}^{k} p_r(n)$ by $P_k(n)$,

$$\frac{P_k(n)}{p(n)} \sim \exp\left(-\frac{2}{c}e^{-\frac{1}{2}cx}\right),$$

* *Proceedings Nat. Acad. Sc.* 23 (1937), 78-84.

† *Duke Math. Jour.* 8 (1941), 335-45.

where $k = \frac{n^{\frac{1}{2}} \log n}{c} + xn^{\frac{1}{2}}$, and x is a fixed number (independent of n). This function $P_k(n)$ is a monotonic function of k, and is hence much easier to study than $p_k(n)$ of this paper. In fact the tables given at the end of this paper suggest that, regarding n as fixed and k as variable, $p_k(n)$ has a unique maximum in the sense that there exists a number k_0 such that

$$\left.\begin{array}{ll} p_k(n) \geqslant p_{k-1}(n) & \text{for } k \leqslant k_0 \\ p_k(n) \leqslant p_{k+1}(n) & \text{for } k \geqslant k_0. \end{array}\right\} \quad (A)$$

and

Generally there appears to be a unique k_0 but sometimes there is a consecutive set of numbers k_0 with the above property, e.g. $p_k(63)$ is maximum for $k = 13$, and $p_k(14)$ is maximum for $k = 4, 5$, the term *maximum* being used in the sense defined by (A). The value of k_0 suggested (but not proved) by the results of this paper, is asymptotic to $c^{-1}n^{\frac{1}{2}} \log n$. In fact the table shows that k_0 differs from $c^{-1}n^{\frac{1}{2}} \log n$ by a quantity which never exceeds one.

The results proved in this paper are:

THEOREM I

$$\frac{n^{\frac{1}{2}} p_k(n)}{p(n)} \sim \exp\left(-\tfrac{1}{2}cx - \frac{2}{c} e^{-\frac{1}{2}cx}\right),$$

where
$$k = c^{-1}n^{\frac{1}{2}} \log n + xn^{\frac{1}{2}}.$$

THEOREM II

If k_1 is the value of k for which $p_k(n)$ is maximum (i.e., $p_k(n) \leqslant p_{k_1}(n)$ if $k \neq k_1$), then for $n > n_0$

$$n^{\frac{1}{2}} < k_1 < \delta n^{\frac{1}{2}} \log n,$$

where δ is any fixed number $> 1/c$.

It is clear that several problems remain open, e.g. the problem of the existence of k_0, and the problem of proving that $k_1 \sim c^{-1}n^{\frac{1}{2}} \log n$.

2. Proof of Theorem II.

We start with the identity

$$p_k(n) = p(n-k) - \sum_{1 \leqslant r \leqslant n-2k} p(n-k-\overline{k+r})$$
$$+ \sum_{\substack{0<r_1<r_2 \\ 1 \leqslant r_1+r_2 \leqslant n-3k}} p(n-k-\overline{k+r_1}-\overline{k+r_2}) - \ldots = p(n)\{S_1 - S_2 + S_3 - \ldots\}.$$

This is, as in EL,* an application of the sieve of Eratosthenes. It is also a consequence of the formula

$$P_k(n) = \sum_{r=1}^{k} p_r(n) = p_k(n+k),$$

applied to the "sieve-formula" for $P_k(n)$ given in EL. This enables us to determine the values of $p_k(n)$ from a table of partitions. We give some simple case

$$p_k(n) = p(n-k) \text{ if } k \geqslant n/2, \quad (B)$$

in all other cases

$$p_k(n) < p(n-k). \quad (B')$$

For $\frac{n^{\frac{1}{2}} \log n}{c'} < k \leqslant n^{\frac{3}{4}}$ where $c' < c$,

$$\frac{p_k(n)}{p(n)} < \frac{p(n-k)}{p(n)} \sim \frac{n}{n-k} \exp\{c(n-k)^{\frac{1}{2}} - cn^{\frac{1}{2}}\}$$
$$\sim \exp\{-\tfrac{1}{2}ck/n^{\frac{1}{2}}\} < n^{-c/2c'},$$

therefore
$$\frac{n^{\frac{1}{2}} p_k(n)}{p(n)} = o(1).$$

From (B') and the Hardy-Ramanujan formula we show that

$$\frac{n^{\frac{1}{2}} p_k(n)}{p(n)} = o(1) \text{ for } k \geqslant n^{\frac{3}{4}}.$$

This is also a consequence of Theorem I. It follows that k_1 of Theorem II satisfies $k_1 < \delta n^{\frac{1}{2}} \log n$, where δ is any constant $> 1/c$, i.e. the second half of Theorem II is proved.

*The paper of Erdos and Lehner will be referred to as EL.

To prove the first half, we use the following inequality,*

$$\frac{1}{k!}\binom{n-1}{k-1} < p_k(n) < \frac{1}{k!}\binom{n-1+k(k-1)/2}{k-1}. \qquad (C)$$

Then $\frac{1}{k!}\binom{n-1}{k-1} = \frac{(n-1)!}{(k-1)!\,(n-k)!\,k!} < \frac{n^k}{(k!)^2}$, whose maximum is asymptotic to $e^{2\sqrt{n}}\,Gn^{\frac{1}{4}}$, where G is a positive constant. For $k \leqslant n^{\frac{1}{2}}$

$$\binom{n-1+k(k-1)/2}{k-1} < \left\{1+\frac{k^2-k}{2(n-k+1)}\right\}^{k-1} < f(n),$$

where $f(n) \sim \exp\left\{n^{\frac{1}{2}}(\log(\ )+\theta)\right\}$, where θ is any fixed positive constant. Hence when $k \leqslant n^{\frac{1}{2}}$, $p_k(n) < g(n)$, where

$$g(n) \sim n^{-\frac{1}{4}} \exp\left\{n^{\frac{1}{2}}(\log\tfrac{\ }{2}+2+\theta)\right\}.$$

But, by taking a suitable value of θ, $\log\frac{\ }{2}+2+\theta < 2.5$, while $\pi\sqrt{\tfrac{2}{3}} > 2.5$. Thus

$$\frac{n^{\frac{1}{2}}\,p_k(n)}{p(n)} = o(1) \text{ for } k \leqslant n^{\frac{1}{2}},$$

which proves Theorem II.

3. PROOF OF THEOREM I. Consider values of k given by

$$k = c^{-1}n^{\frac{1}{2}}\log n + xn^{\frac{1}{2}}.$$

Evaluating $p_k(n)$ with the help of the sieve given in § 2, we get

$$S_1 = \frac{p(n-k)}{p(n)} \sim \frac{1}{n^{\frac{1}{2}}}\exp\left(-\tfrac{1}{2}cx\right).$$

$$S_2 \sim \sum \frac{n}{n-2k-r}\exp\left\{c(n-2k-r)^{\frac{1}{2}}-cn^{\frac{1}{2}}\right\} = \sum_{\substack{1\\ r\leqslant n^{\frac{3}{5}}}} + \sum_{\substack{2\\ r>n^{\frac{3}{5}}}}.$$

In $\sum_1$, $\frac{n}{n-2k-r} \sim 1$ and $(n-2k-r)^{\frac{1}{2}}-n^{\frac{1}{2}} \sim -\frac{1}{2}(2k+r)/n^{\frac{1}{2}}$.

* See H. Gupta, *Proc. Ind. Ac. Sc.* 16 (1942), 101-2, and Auluck in this *Jour.* (current issue) pp. 113-4.

Thus
$$\sum_1 \sim \frac{1}{n}e^{-cx}\sum_{1\leqslant r\leqslant n^{\frac{3}{5}}}\exp\left(-\tfrac{1}{2}crn^{-\frac{1}{2}}\right)$$

$$= n^{-1}e^{-cx}\exp\left(-\tfrac{1}{2}cn^{-\frac{1}{2}}\right)\frac{1-\exp\left(-\frac{1}{2}cn^{\frac{1}{10}}\right)}{1-\exp\left(-\frac{1}{2}cn^{-\frac{1}{2}}\right)}$$

$$\sim 2c^{-1}n^{-1}e^{-cx}n^{\frac{1}{2}}.$$

$$\sum_2 < n\sum_{r>n^{\frac{3}{5}}}\exp\left\{-\tfrac{1}{2}cn^{-\frac{1}{2}}(2k+r)\right\}$$

$$= n.n^{-1}e^{-cx}\sum_{r>n^{\frac{3}{5}}}\exp\left(-\tfrac{1}{2}crn^{-\frac{1}{2}}\right)$$

$$< e^{-cx}\exp\left(-\tfrac{1}{2}cn^{\frac{3}{5}}n^{-\frac{1}{2}}\right)\sum_{r>n^{\frac{3}{5}}} 1$$

$$< ne^{-cx}\exp\left(-\tfrac{1}{2}cn^{\frac{1}{10}}\right)$$
$$= o(1).$$

Therefore
$$S_2 \sim 2c^{-1}n^{-\frac{1}{2}}e^{-cx}.$$

Again
$$S_3 = \frac{1}{2!}\left[\frac{1}{p(n)}\sum_{1\leqslant r_1+r_2\leqslant n-3k} p(n-3k-r_1-r_2)\right.$$

$$\left.-\frac{1}{p(n)}\sum_{1\leqslant 2r\leqslant n-3k} p(n-3k-2r)\right] = \frac{1}{2!}\left[\sum_1 + \sum_2 - \sum_{\substack{3\\ r\leqslant n^{\frac{3}{5}}}} - \sum_{\substack{4\\ r>n^{\frac{3}{5}}}}\right],$$

where Σ_1 runs over all pairs (r_1, r_2) in which neither r_1 nor r_2 exceeds $n^{\frac{3}{5}}$, Σ_2 over all pairs in which at least one number exceeds $n^{\frac{3}{5}}$.

$$\sum_1 \sim n^{-\frac{3}{2}}\exp\left(-\frac{3c}{2}x\right)\sum_{r_1, r_2\leqslant n^{\frac{3}{5}}}\exp\left[-\tfrac{1}{2}cn^{-\frac{1}{2}}(r_1+r_2)\right]$$

$$= n^{-\frac{3}{2}}\exp\left(-\frac{3c}{2}x\right)\left(\sum_{r_1\leqslant n^{\frac{3}{5}}}\exp\left(-\tfrac{1}{2}cr_1n^{-\frac{1}{2}}\right)\right)^2$$

$$\sim n^{-\frac{3}{2}}\exp(-3cx/2)\,4n/c^2$$
$$= n^{-\frac{1}{2}}e^{-\frac{1}{2}cx}\left[2c^{-1}e^{-\frac{1}{2}cx}\right]^2.$$

$$\sum_2 < n^{-\frac{1}{2}} \exp(-3cx/2) \sum_{1 \leqslant r_1 \leqslant n} \exp\left(-\tfrac{1}{2}cr_1 n^{-\frac{1}{2}}\right)$$

$$\times \sum_{r_2 > n^{\frac{3}{5}}} \exp\left(-\tfrac{1}{2}cr_2 n^{-\frac{1}{2}}\right)$$

$$< 2n^{-\frac{1}{2}} e^{-3cx/2} e^{-\frac{1}{2}cn^{-\frac{1}{2}}} c^{-1} n^{\frac{1}{2}} e^{-\frac{1}{2}cn^{1/10}}$$

$$= o(1).$$

$$\sum_3 \sim n^{-\frac{3}{2}} \exp(-3cx/2) \sum_{r \leqslant n^{\frac{3}{5}}} \exp\left(-crn^{-\frac{1}{2}}\right)$$

$$\sim n^{-\frac{3}{2}} \exp(-3cx/2) c^{-1} n^{\frac{1}{2}}$$

$$= o(1).$$

Similarly $\Sigma_4 = o(1)$.

Thus $S_3 \sim \dfrac{1}{2!} \dfrac{1}{n^{\frac{1}{2}}} e^{-\frac{1}{2}cx} \left[\dfrac{2}{c} \exp\left(-\tfrac{1}{2}cx\right)\right]^2$.

. . . .

$$S_{v+1} \sim \frac{1}{v!} \frac{1}{n^{\frac{1}{2}}} e^{-\frac{1}{2}cx} \left[\frac{2}{c} \exp\left(-\tfrac{1}{2}cx\right)\right]^v.$$

But $\quad S_1 - S_2 + \ldots - S_{2v} < \dfrac{p_v(n)}{p(n)} < S_1 - S_2 + \ldots + S_{2v-1}$

and $S_v \to 0$ as $v \to \infty$. Hence

$$\frac{p_k(n)}{p(n)} \sim \sum_{v=1}^{\infty} (-)^{v-1} S_v \sim n^{-\frac{1}{2}} \exp\left[-\tfrac{1}{2}cx - 2c^{-1}e^{-\frac{1}{2}cx}\right],$$

which is Theorem I.

4. The above proof follows the method of EL, without however borrowing any results from that paper. A shorter proof is possible if we use the results of EL combined with the formula

$$P_k(n) = \sum_{r=1}^{k} p_r(n) = p_k(n+k).$$

Thus from EL

$$\frac{p_k(n+k)}{p(n)} \sim \exp\left(-2c^{-1}e^{-cx/2}\right),$$

where $k = c^{-1}n^{\frac{1}{2}} \log n + xn^{\frac{1}{2}}$. Hence changing n into $n-k$ (regarding k as fixed)

$$\frac{p_k(n)}{p(n-k)} \sim \exp\left(-\frac{2}{c} e^{-\frac{c}{2}x_1}\right),$$

where x_1 is defined by

$$k = \frac{(n-k)^{\frac{1}{2}} \log (n-k)}{c} + x_1(n-k)^{\frac{1}{2}}$$

$$= \frac{n^{\frac{1}{2}} \log n}{c} + O\left(\frac{\log n}{\sqrt{n}}\right) + x_1(n-k)^{\frac{1}{2}}.$$

We write the last expression as $c^{-1}n^{\frac{1}{2}} \log n + xn^{\frac{1}{2}}$,

so that $\quad x = x_1\left(\dfrac{n-k}{n}\right)^{\frac{1}{2}} + O\left(\dfrac{\log n}{n}\right).$

But $k = O(n^{\frac{1}{2}} \log n)$ since x is fixed, and hence $\lim_{n\to\infty} x/x_1 = 1$. Hence

$$\frac{p_k(n)}{p(n-k)} = \exp\left(-\frac{2}{c} e^{-\frac{1}{2}cx}\right).$$

where $\quad k = \dfrac{n^{\frac{1}{2}} \log n}{c} + xn^{\frac{1}{2}}.$

It follows that

$$\frac{p_k(n)}{p(n)} \sim \frac{p(n-k)}{p(n)} \exp\left(-\frac{2}{c} e^{-cx/2}\right)$$

$$\sim n^{-\frac{1}{2}} \exp\left(-\tfrac{1}{2}cx - 2c^{-1}e^{-\frac{1}{2}cx}\right),$$

where k is defined above.

5. The following table gives the values of k for which $p_k(n)$ is maximum, and the corresponding values of $p_k(n)$.