L'Ens Math 6 (1960)

# ON THE ZEROS OF BERNOULLI POLYNOMIALS OF EVEN ORDER

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§ 1. It is well known that the Bernoulli polynomial $B_{2\nu}(x)$ has exactly one zero $r_\nu$ between 0 and $\frac{1}{4}$. NÖRLUND [2], p. 131, proves that $r_\nu$ lies between $\frac{1}{6}$ and $\frac{1}{4}$ and states that $r_\nu$ tends to $\frac{1}{4}$ as $\nu \to \infty$.

In what follows we shall prove that $r_\nu$ tends *monotonically* to $\frac{1}{4}$. More precisely we have

$$\frac{\frac{1}{4}-r_1}{\frac{1}{4}-r_2} > 4 > \frac{\frac{1}{4}-r_\nu}{\frac{1}{4}-r_{\nu+1}} > 3 \qquad (\nu = 2, 3, \ldots) . \tag{1}$$

This follows from the relation

$$\theta_\nu \overset{\text{def}}{=} 2\pi\left(\frac{1}{4}-r_\nu\right) = \frac{1}{2^{2\nu}} - \frac{1}{4^{2\nu}} + \frac{4}{6^{2\nu}} - \frac{17}{6}\frac{1}{8^{2\nu}} - \frac{4}{10^{2\nu}} - \frac{4}{12^{2\nu}} + \\ + \frac{13\,p}{14^{2\nu}}, \quad 0 < p < 1 \ (\nu = 1, 2, \ldots) . \tag{2}$$

We obtain (2) from the relation derived in sec. 12 of this paper:

$$\sin\theta_\nu = \frac{1}{2^{2\nu}} - \frac{1}{4^{2\nu}} + \frac{4}{6^{2\nu}} - \frac{3}{8^{2\nu}} - \frac{4}{10^{2\nu}} - \frac{4}{12^{2\nu}} + \\ + \frac{13\,p}{14^{2\nu}}, \quad \frac{1}{40} \leqq p < 1 \quad (\nu = 1, 2, \ldots) . \tag{3}$$

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In 1940 D. H. LEHMER [1] proved that

$$\theta_\nu < \frac{1}{2^{2\nu}} \tag{4}$$

and

$$\theta_\nu = \frac{1}{2^{2\nu}} - \frac{1}{4^{2\nu}} + \frac{4}{6^{2\nu}} + O\left(\frac{1}{8^{2\nu}}\right) \qquad (\nu \to \infty)\,. \tag{5}$$

However, obviously the monotony of the $r_\nu$ cannot be derived from an asymptotic relation like (5), nor can a relation of this type be used for calculating the $r_\nu$. [1])

In the last part of this paper I discuss the asymptotic development of $\sin \theta_\nu$ into a Dirichlet series with even denominators and integer coefficients. Some numerical computations made by Dr. Miller suggested that these coefficients may present a certain interest from the arithmetical point of view. We prove indeed some congruence properties of these coefficients mod 4 and mod 8. Further, certain sequences of these coefficients can be explicitly determined using Riemann's $\zeta$-function and the Dirichlet's L-function corresponding to the modulus 4.

§ 2. We will write from now on $\mu$ for $2\rho$ and use with Lehmer the expression

$$(-1)^{\nu-1} \frac{(2\pi)^\mu}{\mu!} B_\mu(x) = \sum_{x=1}^{\infty} \frac{\cos 2x\pi x}{x^\mu}\,. \tag{6}$$

We have from (2)

$$r_\nu = \frac{1}{4} - \frac{\theta}{2\pi}\,, \tag{7}$$

where we omit for simplicity sake the index $\rho$ of $\theta$. Introducing (7) into (6) we obtain, putting the result $= 0$,

$$\sin\theta + \sum_{\omega=1}^{\infty} (-1)^\omega \frac{\cos 2\omega\theta}{(2\omega)^\mu} + \sum_{\omega=1}^{\infty} (-1)^\omega \frac{\sin(2\omega+1)\theta}{(2\omega+1)^\mu} = 0\,. \tag{8}$$

[1]) Our first proof of the monotony of the $r_\nu$ was obtained in using the Newton-Raphson approximation. It may be mentioned that for the numerical computation of the values $r_\nu$ Dr. Miller found it particularly convenient to derive from the formula (8) in sec. 2 an approximate quadratic equation for $\sin\theta_\nu$ with an error term $O(2^{4-2\nu})$.



Put

$$\sigma = \sin\theta\,, \quad \delta = \frac{1}{2^\mu}\,, \tag{9}$$

then we obtain, isolating in our equation the term $\sin\theta = \sigma$,

$$\sigma < \delta\cos 2\theta + \frac{\sin 3\theta}{3^\mu} - \delta^2\cos 4\theta + \sum_{x=5}^{\infty} \frac{1}{x^\mu}\,. \tag{10}$$

§ 3. We will have to use repeatedly the inequality

$$\sin mx < m\sin x \left(m > 1\,,\ 0 < mx < \frac{3\pi}{2}\right), \tag{11}$$

which follows easily directly for $m = 2$ and for $m \geqq 3$ from the fact that the expression

$$(\sin mx - m\sin x)' = m(\cos mx - \cos x)$$

is negative under the conditions indicated in (11). It follows then, since

$$\theta = 2\pi\left(\frac{1}{4} - r_\nu\right) < \frac{2\pi}{12} = \frac{\pi}{6} < 0.6\,,$$

$$\sin 3\theta < 3\sigma\,, \quad \cos 4\theta = 1 - 2\sin^2 2\theta > 1 - 8\sigma^2\,.$$

Introducing this into (10) and using $\cos 2\theta = 1 - 2\sigma^2$ we obtain

$$\sigma < \delta(1 - 2\sigma^2) + \frac{3\sigma}{3^\mu} - \delta^2(1 - 8\sigma^2) + \sum_{x=5}^{\infty} \frac{1}{x^\mu}\,. \tag{12}$$

We will assume from now on that we have

$$\nu \geqq 5\,, \quad \mu \geqq 10\,. \tag{13}$$

This assumption will be dropped only in the section 12. — Then we have

$$\frac{1}{\delta^2}\sum_{x=5}^{\infty}\frac{1}{x^\mu} = \sum_{x=5}^{\infty}\left(\frac{4}{x}\right)^\mu \leq \sum_{x=5}^{\infty}\left(\frac{4}{x}\right)^6 = 4^6\left(\sum_{x=1}^{\infty}\frac{1}{x^6} - 1 - \frac{1}{2^6} - \frac{1}{3^6} - \frac{1}{4^6}\right)$$

$$= 4^6\left(1.01734306 - 1 - \frac{1}{2^6} - \frac{1}{3^6} - \frac{1}{4^6}\right) < 0.42\,.$$

We introduce this into (12), bring the second term to the left and divide by $\delta$. Thus we obtain

$$\left(1-\frac{3}{3^\mu}\right)\sigma/\delta < 1-\sigma^2(2-8\delta)-0.58\delta < 1-0.58\delta .$$

On the other hand, we have from $\nu = 5$ on

$$3\left(\frac{2}{3}\right)^\mu \leqq \frac{2^{10}}{3^9} = 0.052\ldots < \frac{0.08}{1-\delta/2},$$

$$\frac{3}{3^\mu} < \frac{0.08\delta}{1-\delta/2}, \quad 1-\frac{3}{3^{2\nu}} > 1-\frac{0.08\delta}{1-\delta/2} = \frac{1-0.58\delta}{1-\delta/2},$$

and it follows

$$\frac{\sigma}{\delta} < 1-\delta/2, \quad \sigma < \delta - \frac{\delta^2}{2}. \tag{14}$$

As the arc sin series,

$$\arcsin x = x + \frac{x^3}{6} + \sum_{\nu=2}^{\infty} \frac{1\cdot 3\cdot\ldots\cdot(2\nu-1)}{2\cdot 4\cdot\ldots\cdot(2\nu)} \frac{x^{2\nu+1}}{2\nu+1},$$

has monotonically decreasing positive coefficients, we have easily for $0 < x < 1$ and convenient $p$ with $o < p < 1$:

$$\arcsin x = x + \frac{x^3}{6} + \frac{3px^5}{40}\frac{1}{1-x^2} \tag{15}$$

For $x = \sigma = \sin\theta < \delta = \frac{1}{2^{2\nu}}$ it follows

$$\theta < \sigma\left[1+\frac{\sigma}{6(1-\sigma^2)}\sigma\right] <$$

$$< \delta\left(1-\frac{\delta}{2}\right)\left[1+\frac{\sigma}{2}\right] < \delta\left(1-\frac{\delta}{2}\right)\left(1+\frac{\delta}{2}\right),$$

$$\theta < \delta, \tag{16}$$

which proves Lehmer's inequality $\theta < \delta$ for $\nu \geqq 5$.

§ 4. We rewrite now (8) in the form

$$\sigma = \sum_{\omega=1}^{6}(-1)^{\omega+1}\frac{\cos 2\omega\theta}{(2\omega)^\mu} + \sum_{\omega=1}^{5}(-1)^{\omega+1}\frac{\sin(2\omega+1)\theta}{(2\omega+1)^\mu} + S, \tag{17a}$$

$$S = \sum_{\omega=7}^{\infty}(-1)^{\omega+1}\frac{\cos 2\omega\theta}{(2\omega)^\mu} + \sum_{\omega=6}^{\infty}(-1)^{\omega+1}\frac{\sin(2\omega+1)\theta}{(2\omega+1)^\mu}. \tag{17b}$$



We shall express the remainder terms in multiples of

$$m = 14^{-\mu}. \tag{18}$$

For any integer $n > 14$ we have by (13)

$$n^{-\mu}/m = (14/n)^\mu \leqq (14/n)^{10}$$

and therefore

$$\frac{1}{16^\mu} < 0.27\,m, \ \frac{1}{18^\mu} < 0.082\,m, \ \frac{1}{20^\mu} < 0.032\,m, \ \frac{1}{24^\mu} < 0.005\,m. \tag{19}$$

In our estimates we shall use the inequality

$$\frac{\sin x\alpha}{x^s} > \frac{\sin(x+1)\alpha}{(x+1)^s} \quad (s \geqq 1, \ x > 0, \ (x+1)\alpha < \pi). \tag{20}$$

To prove this inequality, observe that it is sufficient to prove

$$\frac{x+1}{x} > \frac{\sin(x+1)\alpha}{\sin x\alpha},$$

that is,

$$(x+1)\sin x\alpha - x\sin(x+1)\alpha > 0.$$

But the expression to the left is $= 0$ for $\alpha = 0$ and its derivative with respect to $\alpha$,

$$(x+1)\,x\,(\cos x\alpha - \cos(x+1)\alpha)$$

is $> 0$ under the conditions of (20).

§ 5. We use now in the first sum of (17b) the formula

$$\cos 2\omega\theta = 1 - 2\sin^2\omega\theta$$

and decompose S as follows

$$S = \sum_{\omega=7}^{\infty}\frac{(-1)^{\omega+1}}{(2\omega)^\mu} - 2\sum_{\omega=7}^{\infty}(-1)^{\omega+1}\frac{\sin^2\omega\theta}{(2\omega)^\mu} +$$

$$+ \sum_{\omega=6}^{\infty}(-1)^{\omega+1}\frac{\sin(2\omega+1)\theta}{(2\omega+1)^\mu}. \tag{21}$$

Here we have by (18) and (19):

$$m > \sum_{\omega=7}^{\infty} \frac{(-1)^{\omega-1}}{(2\omega)^{\mu}} > m - \frac{1}{16^{\mu}} \geqq m\,(1 - (14/16)^{10}) > (1 - 0.27)\,m\,.$$

In what follows, we will use the letter $p$, with or without indices, to denote positive numbers $< 1$, which need not be otherwise specified and need not be the same.

We can therefore write

$$\sum_{\omega=7}^{\infty} \frac{(-1)^{\omega-1}}{(2\omega)^{\mu}} = (1 - 0.27\,p)\,m\,. \tag{22}$$

Further, we have, using (11) and (14),

$$2\left|\sum_{\omega=7}^{\infty}(-1)^{\omega-1}\frac{\sin^2\omega\theta}{(2\omega)^{\mu}}\right| < \frac{1}{2^{3\mu-1}}\sum_{\omega=7}^{1000}\frac{1}{\omega^{\mu-2}} + \frac{1}{2^{\mu-1}}\sum_{\omega=1001}^{\infty}\frac{1}{\omega^{\mu}} <$$

$$< \frac{1}{2^{3\mu-1}}\int_{6}^{\infty}\frac{dx}{x^{\mu-2}} + \frac{1}{2^{\mu-1}}\int_{1000}^{\infty}\frac{dx}{x^{\mu}} < \frac{432}{7}\,\frac{1}{8^{\mu}\,.\,6^{\mu}} + \frac{2000}{9.2000^{\mu}} <$$

$$< \frac{100}{48^{\mu}} < 100\left(\frac{7}{24}\right)^{10} m < 100\left(\frac{50}{500}\right)^{5} m < 0.01\,m\,.$$

And, again, by (11) and (19)

$$\left|\sum_{\omega=6}^{\infty}(-1)^{\omega+1}\frac{\sin(2\omega+1)\theta}{(2\omega+1)^{\mu}}\right| < \delta\sum_{\omega=6}^{\infty}\frac{1}{(2\omega+1)^{\mu-1}} < \frac{2}{2^{4\nu}}\sum_{\omega=6}^{\infty}\frac{1}{\omega^{\mu-1}} <$$

$$< \frac{2}{4^{\mu}}\int_{5}^{\infty}\frac{dx}{x^{\mu-1}} < \frac{25}{8.4^{\mu}}\cdot\frac{1}{5^{\mu}} = \frac{25}{8}\,\frac{1}{20^{\mu}} < 0.1\,m\,.$$

It follows now from (21) and (22)

$$S = (1 - 0.38\,p)\,m\,. \tag{23}$$

§ 6. We consider the second sum in (17$a$). We have by (14) and (20)

$$\frac{\sin 7\theta}{7^{\mu}} - \frac{\sin 9\theta}{9^{\mu}} + \frac{\sin 11\theta}{11^{\mu}} = p\,\frac{\sin 7\theta}{7^{\mu}} = p'\,\frac{7\sigma}{7^{\mu}} = 7pm\,.$$



Further, using $\sin 3\theta = 3\sigma - 4\sigma^3$, $\sin 5\theta = 5\sigma - 20\sigma^3 + 16\sigma^5$,

$$\frac{\sin 3\theta}{3^{\mu}} - \frac{\sin 5\theta}{5^{\mu}}\left(\frac{3}{3^{\mu}} - \frac{5}{5^{\mu}}\right)\sigma = -4\sigma^3\left[\frac{1}{3^{\mu}} - \frac{5}{5^{\mu}} + 4\,\frac{\sigma^2}{5^{\mu}}\right] =$$

$$= -\frac{4p}{24^{\mu}} = -0.02\,pm\,.$$

Therefore

$$\sum_{\omega=1}^{5}(-1)^{\omega+1}\frac{\sin(2\omega+1)\theta}{(2\omega+1)^{\mu}} = \left(\frac{3}{3^{\mu}} - \frac{5}{5^{\mu}}\right)\sigma + (7p - 0.02p')\,m\,. \tag{24}$$

§ 7. Consider now the first sum in (17$a$). We have

$$\sum_{\omega=1}^{6}(-1)^{\omega+1}\frac{\cos 2\omega\theta}{(2\omega)^{\mu}} - \left(\frac{1}{2^{\mu}} - \frac{1}{4^{\mu}} + \frac{1}{6^{\mu}} - \frac{1}{8^{\mu}} + \frac{1}{10^{\mu}} - \frac{1}{12^{\mu}}\right) =$$

$$= -2\left(\frac{\sigma^2}{2^{\mu}} - \frac{\sin^2 2\theta}{4^{\mu}} + \frac{\sin^2 3\theta}{6^{\mu}} - \frac{\sin^2 4\theta}{8^{\mu}} + \frac{\sin^2 5\theta}{10^{\mu}} - \frac{\sin^2 6\theta}{12^{\mu}}\right) =$$

$$= -2\delta\sigma^2 + S_1\,,$$

$$S_1 = 2\sum_{\omega=2}^{6}(-1)^{\omega}\frac{\sin^2\omega\theta}{(2\omega)^{\mu}}\,,$$

where by (20) and (19)

$$S_1 = 2p\,\frac{\sin^2 2\theta}{4^{\mu}} = \frac{8p'\sigma^2}{4^{\mu}} = \frac{8p''}{16^{\mu}} = 2.2pm\,.$$

We have, therefore,

$$\sum_{\omega=1}^{6}(-1)^{\omega+1}\frac{\cos 2\omega\theta}{(2\omega)^{\mu}} = \sum_{\omega=1}^{6}\frac{(-1)^{\omega+1}}{(2\omega)^{\mu}} - 2\delta\sigma^2 + 2.2\,pm\,. \tag{25}$$

We have now by (23), (24) and (25) from (17$a$)

$$\sigma = \sum_{\omega=1}^{6}\frac{(-1)^{\omega+1}}{(2\omega)^{\mu}} + \left(\frac{3}{3^{\mu}} - \frac{5}{5^{\mu}}\right)\sigma - 2\delta\sigma^2 + (1 - 0.4p + 9.2p')m\,. \tag{26}$$

Here the last term can be written as

$$m\,(0.6 + 9.2p + 0.4\,(1 - p')) = (0.6 + 9.6p'')\,m\,.$$

§ 8. We put now, using (14),

$$\sigma = \delta - \eta\,, \quad 0 < \eta < \delta\,. \tag{27}$$

Then, (26) becomes

$$\delta - \eta = \sum_{\omega=1}^{6} \frac{(-1^{\omega-1}}{(2\omega)^{\mu}} + \frac{3}{6^{\mu}} - \frac{5}{10^{\mu}} - \left(\frac{3}{3^{\mu}} - \frac{5}{5^{\mu}}\right)\eta - 2\delta(\delta^2 - 2\delta\eta + \eta^2) + \\ + (0.6 + 9.6p)\,m\,.$$

Here we bring all terms containing $\eta$ to the right:

$$\delta + \sum_{\omega=1}^{6} \frac{(-1)^{\omega}}{(2\omega)^{\mu}} - \frac{3}{6^{\mu}} + \frac{5}{10^{\mu}} + 2\delta^3 - (0.6 + 9.6p)\,m = \\ = \eta\left(1 - \frac{3}{3^{\mu}} + \frac{5}{5^{\mu}}\right) + 4\delta^2\eta - 2\delta\eta^2\,.$$

We put now

$$x = \frac{3}{3^{\mu}} - \frac{4}{4^{\mu}} - \frac{5}{5^{\mu}}\,. \tag{28}$$

Then we obtain

$$(1 - x)\,\eta - 2\delta\eta^2 = \frac{1}{4^{\mu}} - \mathrm{R} + \mathrm{E}\,, \tag{29}$$

where we have put

$$\mathrm{R} = \frac{4}{6^{\mu}} - \frac{3}{8^{\mu}} - \frac{4}{10^{\mu}} - \frac{1}{12^{\mu}}\,, \tag{30}$$

$$\mathrm{E} = -(0.6 + 9.6p)\,m\,. \tag{31}$$

§ 9. The expression $x$ defined by (28) is *positive* as follows from

$$1 = \frac{1}{2} + \frac{1}{2} > \left(\frac{3}{4}\right)^{9} + \left(\frac{3}{5}\right)^{9} \geqq \left(\frac{3}{4}\right)^{\mu-1} + \left(\frac{3}{5}\right)^{\mu-1}\,.$$

In the same way, we see that R is *positive*:

$$4 > 1 + 1 + 1 > 3\left(\frac{3}{4}\right)^{19} + 4\left(\frac{3}{5}\right)^{10} + \left(\frac{1}{2}\right)^{10}\,.$$



We have, therefore,

$$0 < x < \frac{3}{3^{\mu}} < \frac{1}{1001}\,, \quad \frac{1}{1-x} < 1 + \frac{1}{1000}\,,$$

$$\mathrm{R} < \frac{4}{6^{\mu}}\,, \quad x\mathrm{R} = \frac{12p}{18^{\mu}} = 12.0.082\,p'm = 0.99\,p''m\,,$$

$$\frac{x\mathrm{R}}{1-x} = pm\,. \tag{32}$$

Further, by (19),

$$\delta^2 x^2 = \frac{9}{36^{\mu}} \leqq 9.2^{-10}\,(18)^{-\mu} = 0.01\,pm\,,$$

$$\frac{\delta^2 x^2}{1-x} = 0.02\,pm\,. \tag{33}$$

The expression to the right in (29) is, as $\mathrm{R} > 0$, $< \delta^2$, and we have from (29) by (27)

$$\left(1 - \frac{1}{1001}\right)\frac{\eta}{\delta^2} < 2\delta\left(\frac{\eta}{\delta}\right)^2 + 1 < 1.002\,, \quad \eta < 1.04\,\delta^2\,,$$

$$\eta^2 < 1.1\,\delta^4\,, \quad 2\delta\eta^2 < 3\delta^5 = 3\left(\frac{7}{16}\right)^{10} pm = 0.1\,(1 - p')\,m\,.$$

Using this in (29), we obtain now

$$(1 - x)\,\eta = \delta^2 - \mathrm{R} + \mathrm{E}_1\,, \tag{34}$$

$$\mathrm{E}_1 = -(0.5 + 9.7p)\,m\,. \tag{35}$$

§ 10. From (35), we have now easily

$$|\,\mathrm{E}_1\,| < 11\,m\,,$$

$$\left|\frac{\mathrm{E}_1 x}{1-x}\right| < 0.05m\,. \tag{36}$$

Dividing now (34) by $1 - x$ we have

$$\eta - \delta^2 + \mathrm{R} - \delta^2 x = \frac{\delta^2 x^2}{1-x} - \frac{\mathrm{R}x}{1-x} + \mathrm{E}_1 + \frac{\mathrm{E}_1 x}{1-x}\,, \tag{37}$$

and from (32), (33), (35) and (36), we have for the right-hand expression in (37):

$$(0.02p_1 - p_2 + 0.05p_3 - 0.5 - 9.7p_5 - 0.05p_4) = -(0.4 + 10.9)\,pm.$$

On the other hand, we have from (28) by (19)

$$\delta^2 x - \frac{3}{12^\mu} = -\left(\frac{4}{16^\mu} + \frac{5}{20^\mu}\right) = -\left(4\left(\frac{7}{8}\right)^{10} + 5\left(\frac{7}{10}\right)^{10}\right) pm\,;$$

$$\delta^2 x = \frac{3}{12^\mu} - 1.4\,pm\,.$$

Introducing this into (37), we finally obtain

$$\eta_\nu = \frac{1}{4^\mu} - \frac{4}{6^\mu} + \frac{3}{8^\mu} + \frac{4}{16^\mu} + \frac{4}{12^\mu} - \frac{0.4 + 12.3p}{14^\mu}\,. \qquad (38)$$

By (27), we have then

$$\sigma = \sin\theta = \frac{1}{2^\mu} - \frac{1}{4^\mu} + \frac{4}{6^\mu} - \frac{3}{8^\mu} - \frac{4}{10^\mu} - \frac{4}{12^\mu} + \frac{13p}{14^\mu}\,, \qquad (39)$$

$$\frac{1}{40} \leqq p < 1 \ (\nu \geqq 5).$$

From (38), it now follows easily that

$$\eta < \delta^2\,. \qquad (40)$$

§ 11. We shall apply now (15) to $x = \sigma$. We have obviously by (14) and (19)

$$\frac{\sigma^5}{1-\sigma^2} < \frac{\delta^5}{1-\delta^2} < \frac{1}{16^\mu}\,\frac{1}{1-10^{-6}}\cdot\frac{1}{2^\mu} < \frac{0.001}{16^\mu} < \frac{2.7}{10^4}\,m\,,$$

$$\frac{3}{40}\,\frac{\sigma^5}{1-\sigma^2} < \frac{3\,m}{10^5}\,.$$

Further, from (27), (40) and (19)

$$\delta^3 - \sigma^3 = 3\,\delta^2\eta_\nu - 3\,\delta\eta^2 + \eta^3 < 3\,\delta^2\eta < 3\,\delta^4 < 0.9\,m\,,$$

and, therefore, from (15) and (39), as $\frac{0.9}{6} < \frac{13}{40}$,

$$\theta = \frac{1}{2^\mu} - \frac{1}{4^\mu} + \frac{4}{6^\mu} - \frac{17}{6}\,\frac{1}{8^\mu} - \frac{4}{10^\mu} - \frac{4}{12^\mu} + \frac{13p}{14^\mu}\,. \qquad (41)$$

§ 12. The formulae (39) and (41) have been only derived for $\nu \geq 5$. However, the direct comparison of the expressions



given by these formulae with the values of $r_\nu$ for $\nu = 1, 2, 3, 4$ given with ten decimals by Lehmer [1]) shows that these formulae also hold for $\nu = 1, 2, 3, 4$. This proves the formulae (2) and (3).

§ 13. Lehmer's formula (5) and our formulae (2), (3) suggest that $\theta$, as well as $\sin\theta$, possess asymptotic development by infinite Dirichlet series. In this connection, apparently, the development of $\sin\theta$ gives a more natural and more interesting result. We will now prove that there exists a Dirichlet series

$$\sum_{n=1}^{\infty} \frac{q_n}{(2n)^\mu} \qquad (42)$$

with *integer* $q_n$ such that we have for $\nu \to \infty$ and every positive integer N

$$\eta_N \overset{\text{def}}{=} \sin\theta_\nu - \sum_{n=1}^{N} \frac{q_n}{(2n)^\mu} = O\left((2N+2)^{-\mu}\right)(\nu \to \infty)\,\ldots \qquad (43)$$

§ 14. It is obvious from (3) that (43) is true for $N \leq 6$ for a certain Dirichlet polynomial. Assume that we have for a certain value N the relation (43) for a certain Dirichlet polynomial with N terms. Introducing (8) into (43), we have then

$$\eta_N = \sum_{\omega=1}^{2N} (-1)^{\omega+1}\left(\frac{\cos 2\omega\theta_\nu}{(2\omega)^\mu} + \frac{\sin(2\omega+1)\,\theta_\nu}{(2\omega+1)^\mu}\right) - \sum_{\omega=1}^{N}\frac{q_\omega}{(2\omega)^\mu} + O\left(\frac{1}{(4N+2)^\mu}\right). \qquad (44)$$

Here both $\cos 2\omega\theta$ and $\sin(2\omega+1)\,\theta$ are polynomials in $\sigma_\nu = \sin\theta_\nu$ *with integer coefficients*. Putting these polynomials in (44), we obtain then, denoting by M the greatest of their degrees,

$$\cos 2\omega\theta_\nu = \sum_{\varkappa=0}^{M} a_\varkappa\,\sigma_\nu^\varkappa = \sum_{n=1}^{2N}\frac{A_n^{(\omega)}}{(2n)^\mu} + O\left(\frac{1}{(4N+2)^\mu}\right),$$

[1]) These values have been checked independently of Lehmer by Dr. J. C. P. Miller.

$$\sin(2\omega+1)\theta_\nu = \sum_{\varkappa=1}^{M} b_\varkappa \sigma_\nu^\varkappa = \sum_{n=1}^{2N} \frac{B_n^{(\omega)}}{(2n)^\mu} + O\left(\frac{1}{(4N+2)^\mu}\right),$$

$$\frac{\cos 2\omega\theta_\nu}{(2\omega)^\mu} - \frac{\sin(2\omega+1)\theta_\nu}{(2\omega+1)^\mu} = \sum_{n=1}^{2N} \frac{C_n^{(\omega)}}{(2n)^\mu} + O\left(\frac{1}{(4N+2)^\mu}\right),$$

with integers $A_n^{(\omega)}$, $B_n^{(\omega)}$ and $C_n^{(\omega)}$, and therefore

$$\tau_N = \sum_{n=1}^{2N} \frac{D_n}{(2n)^\mu} + O\left(\frac{1}{(4N+2)^\mu}\right) \tag{45}$$

with integer $D_N$. But, now it follows from (43) that

$$D_1 = D_2 = \ldots = D_N = 0,$$

and we have, therefore, putting

$$q_n = D_n \ (N < n \leqq 2N), \tag{46}$$

again the relation (43) with 2N instead of N.

Repeating this procedure indefinitely, the existence of the Dirichlet series (42) is proved.

Introducing the asymptotic development (42) into the Maclaurin series for arcsin $x$, we obtain an asymptotic Dirichlet development of $\theta_\nu$ itself. However, in this development, as we see from (2), the coefficients are no longer integers.

§ 15. In what follows, we give with the kind permission of Dr. J. C. P. Miller the first fifty coefficients of the series (42) which he has computed.

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| $q_n$ | 1 | −1 | 4 | −3 | −4 | −4 | 8 | 11 | 4 | 4 |
| $n$ | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
| $q_n$ | 12 | −48 | −12 | −8 | −16 | 25 | −16 | −4 | 20 | 0 |
| $n$ | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 |
| $q_n$ | 32 | −12 | 24 | 248 | −4 | 12 | 4 | −208 | −28 | 16 |
| $n$ | 31 | 32 | 33 | 34 | 35 | 36 | 37 | 38 | 39 | 40 |
| $q_n$ | 32 | −41 | 48 | 16 | −32 | −400 | −36 | −20 | −48 | 88 |
| $n$ | 41 | 42 | 43 | 44 | 45 | 46 | 47 | 48 | 49 | 50 |
| $q_n$ | −40 | −32 | 44 | −544 | −16 | −24 | 48 | 732 | 8 | 4 |



Dr. Miller drew my attention to the properties of the coefficients of the series (42) which appear to be suggested by the above values. The odd coefficients correspond exactly to the denominators which are powers of 2, while all other numerators are divisible by 4.

We will now prove that these properties are indeed true in the general case. Beyond that, we will prove for the numerators $q_n$ which correspond to $n = 2^k$, that we have

$$q_n \equiv (-1)^k \equiv 2k+1 \pmod 4 \quad (n = 2^k). \tag{47}$$

We will prove even a more precise formula

$$q_n \equiv (-1)^{k-1} 2(k-1) - 1 \equiv 4k^2 + 2k + 1 \pmod 8 \ (n = 2^k). \tag{48}$$

Further, we will determine directly all $q_n$ corresponding to $n$ non divisible by 8 by forming generating Dirichlet series for these numerators. We will find in particular for an odd natural $u$

$$q_{2u} = -q_u \ (u \equiv 1 \pmod 2). \tag{49}$$

§ 16. Expressing $\sin(2\omega+1)\theta$ and $\cos 2\omega\theta$ as polynomials in $\sigma = \sin\theta$, we have, putting $2\omega+1 = u$, $2\omega = g$,

$$\sin u\theta = u\sigma + R_n(\sigma), \tag{50}$$

$$\cos g\theta - 1 = -g^2\frac{\sigma^2}{2} + T_g(\sigma), \tag{51}$$

where our $R_u(\sigma)$ and $T_g(\sigma)$ are polynomials in $\sigma$ with integer coefficients, which are all *multiples of* 4.

Indeed, this is true for $\omega = 1$. Assuming our assertions true for a certain $\omega$, use the following relations and congruences mod 4:

$$\cos(2\omega+2)\theta - \cos 2\omega\theta = -2\sin\theta\sin(2\omega+1)\theta \equiv -2\sigma(2\omega+1)\sigma \equiv -2\sigma^2 \pmod 4,$$

$$\cos(2\omega+2)\theta \equiv 1 - 2\omega^2\sigma^2 - 2\sigma^2 \equiv 1 - 2(\omega+1)^2\sigma^2 \pmod 4,$$

which proves our assertion (51) for $\omega+1$, and

$$\sin(2\omega+3)\theta - \sin(2\omega+1)\theta = 2\sigma\cos 2(\omega+1)\theta \equiv 2\sigma(1 - 2(\omega+1)^2\sigma^2) \equiv 2\sigma \pmod 4,$$

which proves our assertion (50) for $\omega + 1$.

Further, it follows from the identity with a natural $k$:

$$\sin 2^k \theta = 2^k \sin \theta \prod_{\nu=1}^{k} \cos 2^{k-\nu} \theta ,$$

that, developing $\cos 2^{k+1} \theta$ in powers of $\sigma = \sin \theta$, we have

$$\cos 2^{k+1} \theta = 1 - 2^{2k-1} \sigma^2 \prod_{\nu=1}^{k} \cos^2 2^{k-\nu} \theta \equiv 1 \pmod{2^{2k+1}} . \quad (51^0)$$

§ 17. Introducing the relations (50) and (51) into (8) and solving with respect to $\sigma = \sin \theta$, we obtain

$$\sigma = \sum_{\omega=1}^{\infty} \frac{(-1)^{\omega+1}}{(2\omega)^{\mu}} - \sigma \sum_{\omega=1}^{\infty} \frac{(-1)^{\omega}}{(2\omega+1)^{\mu-1}} + 2\sigma^2 \sum_{\omega=1}^{\infty} \frac{(-1)^{\omega} \omega^2}{(2\omega)^{\mu}} -$$
$$- \sum_{\omega=1}^{\infty} (-1)^{\omega} \left( \frac{T_{2\omega}(\sigma)}{(2\omega)^{\mu}} + \frac{R_{2\omega+1}(\sigma)}{(2\omega+1)^{\mu}} \right) . \quad (52)$$

We introduce now the two Dirichlet series well known in the analytical theory of numbers of which the first is not very different from Riemann's $\zeta$-function while the second is Dirichlet's L-function corresponding to the modulus 4:

$$U(s) = \left(1 - \frac{1}{2^s}\right) \zeta(s) = \sum_u \frac{1}{u^s} , \quad (53)$$

where the summation index $u$ here and in what follows runs through all positive odd integers subject to the restrictions explicitly indicated, and

$$L(s) = \sum_u \frac{(-1)^{\frac{u-1}{2}}}{u^s} = \prod_{p>2} \frac{1}{1 - \frac{(-1)^{\frac{p-1}{2}}}{p^s}} , \quad (54)$$

where $p$ runs through all odd primes. Then we have for the first three right hand terms in (52), putting

$$x = \frac{1}{2^{\mu}} : \quad (55)$$



$$\sum_{\omega=1}^{\infty} \frac{(-1)^{\omega+1}}{(2\omega)^{\mu}} = x U(\mu) - \sum_{\varkappa=2}^{\infty} x^{\varkappa} U(\mu) = \frac{x - 2x^2}{1-x} U(\mu) , \quad (56)$$

$$\sum_{\omega=1}^{\infty} \frac{(-1)^{\omega}}{(2\omega+1)^{\mu-1}} = L(\mu - 1) - 1 , \quad (57)$$

$$\sum_{\omega=1}^{\infty} \frac{(-1)^{\omega+1} \omega^2}{(2\omega)^{\mu}} = \frac{1}{4} \sum_{\omega=1}^{\infty} \frac{(-1)^{\omega+1}}{(2\omega)^{\mu-2}} =$$
$$= x U(\mu - 2) - \frac{4x^2}{1-4x} U(\mu - 2) = \frac{x - 8x^2}{1-4x} U(\mu - 2) . \quad (58)$$

Introducing these expressions into (52) and bringing the term with $\sigma$ to the left, we have

$$L(\mu - 1)\sigma = \frac{x - 2x^2}{1-x} U(\mu) - 2\sigma^2 \frac{x - 8x^2}{1-4x} U(\mu - 2) -$$
$$- \sum_{\omega=1}^{\infty} (-1)^{\omega} \left( \frac{T_{2\omega}(\sigma)}{(2\omega)^{\mu}} + \frac{R_{2\omega+1}(\sigma)}{(2\omega+1)^{\mu}} \right) . \quad (59)$$

§ 18. We can replace $\sigma$ in (59) by the series (42) and consider now (59) modulo 4, taking a Dirichlet series $\sum_{n=1} \frac{a_n}{n^{\mu}}$ as $\equiv 0 \pmod 4$ if all $a_n$ are $\equiv 0 \pmod 4$. Then the last sum in (59) is $\equiv 0$, by what has been proved in Section 16 about (50) and (51). Further, we have mod 4:

$$L(\mu - 1) = \sum_u \frac{(-1)^{\frac{u-1}{2}} u}{u^{\mu}} \equiv \sum_u \frac{1}{u^{\mu}} = U(\mu) ,$$

$$U(\mu - 2) = \sum \frac{(2\omega+1)^2}{(2\omega+1)^{\mu}} \equiv U(\mu) \equiv L(\mu - 1) \pmod 4 .$$

It follows, therefore, from (59)

$$L(\mu - 1)\sigma \equiv \left( \frac{x - 2x^2}{1-x} - 2x\sigma^2 \right) L(\mu - 1) \pmod 4 .$$

If we multiply this congruence on both sides by

$$L^{-1}(\mu - 1) = \prod_{p>2} \left( 1 - \frac{(-1)^{\frac{p-1}{2}} p}{p^{\mu}} \right) ,$$