§

Jordan, Finite Differens 2457

pp 448-450

equation (1) it remains still to compute $\Sigma U_m(x)f(x)$. For this, let us start from the expression (7, § 139) of $U_m(x)$; this will give

$$\sum_{x=a}^{b} U_m(x)f(x) = Ch^{2m} \sum_{\nu=0}^{m+1} \binom{m+\nu}{m} \binom{m-N}{m-\nu} \sum_{\xi=0}^{N} \binom{\xi}{\nu} f(a+\xi).$$

According to § 136 the last sum in the second member is equal to the binomial moment of order ν, denoted by $\mathscr{B}_\nu$, of the function $f(a+\xi h)$; therefore this may be written:

Therefore

$$\sum_{x=a}^{b} U_m(x)f(x) = Ch^{2m} \sum_{\nu=0}^{m+1} \binom{m+\nu}{m} \binom{m-N}{m-\nu} \mathscr{B}_\nu . \tag{3}$$

As will be shown later, there is a far better method for rapidly computing the *binomial moments* than is available in the case of power moments. If we operate with equidistant discontinuous variables, it is not advantageous to consider powers; it is much better to express the quantities by binomial coefficients. Indeed, if an expression were given in power series, it would still be advantageous to transform it into a binomial series.

Several statisticians have remarked that it is not advisable to introduce moments of higher order into the calculations. In fact if N is large, these numbers will increase rapidly with the order of the moments, will become very large, and their coefficients in the formulae will necessarily become very small. It is difficult to operate with such numbers, the causes of errors being many.

To remedy this inconvenience, the *mean binomial moment* has been introduced. The definition of the mean binomial moment $\mathscr{T}_\nu$ of order ν of the function $f(x+\xi h)$ is the following

$$\mathscr{T}_\nu = \sum_{\xi=0}^{N} \binom{\xi}{\nu} f(a+\xi h) \Big/ \sum_{\xi=0}^{N} \binom{\xi}{\nu}$$

therefore

$$\mathscr{T}_\nu = \frac{\mathscr{B}_\nu}{\binom{N}{\nu+1}} . \tag{4}$$

The mean binomial moment will remain of the same order of magnitude as $f(x)$, whatever N or ν may be. For instance, if

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$f(x)$ is equal to the constant k then we shall have $\mathscr{T}_\nu = k$ for any value of ν or N. On the other hand the power moment of order ν

$$k \sum_{\xi=0}^{N} \xi^\nu$$

will increase rapidly with ν and N.

Introducing into formula (3) $\mathscr{T}_\nu$ instead of $\mathscr{B}_\nu$ we shall have

$$\sum_{x=a}^{b} U_m(x) f(x) = Ch^{2m} \sum_{\nu=0}^{m+1} \binom{m+\nu}{m} \binom{m-N}{m-\nu} \binom{N}{\nu+1} \mathscr{T}_\nu .$$

This may be written in the following form

$$(-1)^m Ch^{2m} (m+1) \binom{N}{m+1} \sum_{\nu=0}^{m+1} (-1)^\nu \binom{m+\nu}{m} \binom{m}{\nu} \frac{\mathscr{T}_\nu}{\nu+1} .$$

To simplify the formula we shall write

$$\beta_{m\nu} = (-1)^{m+\nu} \binom{m+\nu}{m} \binom{m}{\nu} \frac{1}{\nu+1} . \tag{5}$$

Since these numbers are very useful they are presented in the following table, which gives all the numbers necessary for parabolas up to the tenth degree.

Table for $\beta_{m\nu}$

$m \backslash \nu$	0	1	2	3	4	5
1	—1	1				
2	1	—3	2			
3	—1	6	—10	5		
4	1	—10	30	—35	14	
5	—1	15	—70	140	—126	42
6	1	—21	140	—420	630	—462
7	—1	28	—252	1050	—2310	2772
8	1	—36	420	—2310	6930	—12012
9	—1	45	—660	4620	—18018	42042
10	1	—55	990	—8580	42042	—126126

2457

$m \diagdown \nu$	6	7	8	9	10
6	132				
7	—1716	429			
8	12012	—6435	1430		
9	—60060	51480	—24310	4862	
10	240240	—291720	218790	—92378	16796

The following relation can be used for checking the numbers:

$$\beta_{m0} + \beta_{m1} + \beta_{m2} + \ldots + \beta_{mm} = 0$$

that is, the sum of the numbers in the rows is equal to zero.

Moreover let us put

$$(6) \qquad \sum_{\nu=0}^{m+1} \beta_{m\nu} \mathcal{T}_\nu = \Theta_m .$$

If we already know the mean binomial moments, the value of Θ_m may readily be computed with the aid of the table above. Finally we obtain

$$(7) \qquad \sum_{x=a}^{b} U_m(x)\, f(x) = Ch^{2m}\,(m+1)\binom{N}{m+1}\,\Theta_m .$$

As this expression could be termed the orthogonal moment of degree m of $f(x)$, therefore we can consider Θ_m as a certain *mean orthogonal moment* of degree m of $f(x)$.

The mean orthogonal moments are independent of the origin of the interval, and of the constant C. Particular case:

$$\Theta_0 = \mathcal{T}_0 = \mathcal{B}_0/N$$

is equal to the arithmetic mean of the quantities $f(x_i)$.

By aid of equation (7) and of (12), § 139 we deduce from (1) the coefficient c_m:

$$(8) \qquad c_m = \frac{(2m+1)\,\Theta_m}{Ch^{2m}\binom{N+m}{m}} .$$

The coefficient c_m is independent of the origin. In particular we have

$$c_0 = \Theta_0/C .$$