Irrationality of the $\zeta$ Function on Odd Integers

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Abstract
The $\zeta$ function is defined by $\zeta(s) = \sum_n 1/n^s$. This talk is a study of the irrationality of the zeta function on odd integer values $> 2$.

1. Introduction

The sum $\sum_n 1/n^2$ was first studied by Bernoulli, who proved around 1680 that it converged to a finite limit less than 2. Euler proved in 1735 that it is equal to $\pi^2/6$, and studied the more general function $\zeta(s) = \sum_n 1/n^s$. He also showed that on even integers the $\zeta$ function has a closed form, namely $\zeta(2n) = C_n \pi^{2n}$ where the coefficients $C_n$ are rational numbers that he wrote in terms of Bernoulli numbers. A century later Riemann studied this function on the whole complex plane, and he stated a conjecture on the location of the zeroes of the zeta function, that is known as the Riemann hypothesis, and is still unproved.

The first result on the irrationality of the $\zeta$ function on odd integers is due to Apéry, who proved in 1978 that $\zeta(3)$ is irrational [1]. Recently Tanguy Rivoal showed that the $\zeta$ function takes infinitely many irrational values on the odd integers [4, 5], and that there exists an odd integer $j$ with $5 \leq j \leq 21$ such that $\zeta(j)$ is irrational [5]. Zudilin [6] refined this result and proved it for $5 \leq j \leq 11$.

2. Irrationality of $\zeta(3)$

Theorem 1 (Apéry(1978)). The number $\zeta(3)$ is irrational.

The following proof is due to Nesterenko [3], after ideas by Beukers. The theorem is proved using the following generating function

$$S_n(z) = \sum_{k=1}^{\infty} \frac{\partial}{\partial k} \left( \frac{(k-1)^2(k-2)^2 \cdots (k-n)^2}{k^2(k+1)^2 \cdots (k+n)^2} \right) z^{-k}$$

The decomposition of the coefficient of $z^{-k}$ in partial fractions gives the equality

$$S_n(z) = P_{0,n}(z) + P_{1,n}(z) \text{Li}_2(1/z) + P_{2,n}(z) \text{Li}_3(1/z)$$

where $\text{Li}_s(z) = \sum_{n \geq 1} z^n/n^s$ is a polylogarithm function, and $P_{k,n}$ are polynomials of degree $n$ such that $P_{1,n}(1) = 0$. When Equation (1) is specialized at $z = 1$, it becomes

$$S_n(1) = P_{0,n}(1) + P_{2,n}(1)\zeta(3),$$

with the additonal properties that $P_{2,n}(1) \in \mathbb{Z}$ and $d_n^2 P_{0,n}(1) \in \mathbb{Z}$ where $d_n = \text{ppcm}(1,2,\ldots,n)$. 

The value $S_n(1)$ is bounded by using an integral representation.

\[ S_n(1) = \frac{1}{2\pi i} \int_L \left( \frac{\Gamma(n+1-s)\Gamma(s)^2}{\Gamma(n+1+s)} \right)^2 ds, \]

where $L$ is the vertical line $\Re(z) = c$, $0 < c < n + 1$, oriented from top to bottom. From this integral, the bounds $0 < S_n(1) \leq c(\sqrt{2} - 1)^{|n|}$ are obtained.

The inequalities $0 < d_n^3 P_{0,n}(1) + d_n^3 P_{2,n}(1) \zeta(3) < cr^n$, where $c$ is a constant, and $r < 1$ prove that $\zeta(3)$ is irrational; because if $\zeta(3)$ is rational and equal to $p/q$, then $qd_n^3 P_{0,n}(1) + qd_n^3 P_{2,n}(1) \zeta(3)$ is an integer greater than 0 and bounded by $qcr^n$ that converges to 0.

3. The $\zeta$ Function Has Infinitely Many Irrational Values on Odd Integers

Tanguy Rivoal in fact proved a stronger result, that is:

**Theorem 2.** Let $a$ be an odd integer greater than 3 and $\delta(a)$ be the dimension of the $\mathbb{Q}$-vector space spanned by $1$, $\zeta(3)$, $\ldots$, $\zeta(a)$, then

\[ \delta(a) \geq \frac{1}{3} \log a. \]

This implies directly that infinitely many $\zeta(2n+1)$ are irrational.

To prove Theorem 2, we introduce the series

\[ S_{n,a,r}(z) = n!^{a-2r} \sum_{k=1}^{\infty} \frac{(k-rn)rn(k+n+1)rn}{(k)_n^{a+1}} z^{-k}, \]

where $(k)_n = k(k+1)\ldots(k+n-1)$ is the Pochhammer symbol, and $n$, $r$, and $a$ are integers satisfying $n \geq 0$, $1 \leq r < a/2$, so that $S_{n,a,r}(z)$ exists when $|z| \geq 1$. As for the proof of the irrationality of $\zeta(3)$, an equality between the series studied and values of $\zeta$ is found, namely

\[ S_{n,a,r}(1) = P_{0,n}(1) + \sum_{l=2}^{a} P_{l,n}(1) \zeta(l), \]

moreover, if $(n+1)a + l$ is odd then $P_{l,n}(1) = 0$. For $n$ odd and $a$ odd greater than 3, $P_{l,n}(1) = 0$ if $l$ is even, so that $S_{n,a,r}(1)$ is a linear combination of values of $\zeta$ on odd integers.

The dimension of the vector space spanned by $1$, $\zeta(3)$, $\ldots$, $\zeta(a)$ is based on the following theorem:

**Theorem 3** (Nesterenko’s criterion). Let $\theta_1$, $\theta_2$, $\ldots$, $\theta_N$ be $N$ real numbers, and suppose that there exist $N$ sequences $(p_{l,n})_{n \geq 0}$ such that

1. $\forall i = 1, \ldots, N$, $p_{l,n} \in \mathbb{Z}$;
2. $\alpha_1^{n+\omega(n)} \leq |\sum_{l=1}^{N} p_{l,n} \theta_l| \leq \alpha_2^{n+\omega(n)}$, with $0 < \alpha_1 \leq \alpha_2 < 1$;
3. $\forall l = 1, \ldots, N$, $|p_{l,n}| \leq \beta^{n+\omega(n)}$ with $\beta > 1$.

Then

\[ \dim_{\mathbb{Q}}(\mathbb{Q} \theta_1 + \mathbb{Q} \theta_2 + \cdots + \mathbb{Q} \theta_N) \geq \frac{\log(\beta) - \log(\alpha_1)}{\log(\beta) - \log(\alpha_1) + \log(\alpha_2)}. \]

This criterion, applied to the real numbers $\theta_i = \zeta(2i+1)$, $i \leq (a - 1)/2$, with the sequences $p_{l,n}$ defined by $p_{0,n} = d_{2n}^3 P_{0,2n}(1)$ and $p_{l,n} = d_{2n}^3 P_{2l+1,2n}(1)$ if $1 \leq l \leq (a - 1)/2$ yields the inequality

\[ \delta(a) \geq \frac{\log(r) + \frac{a-r}{a+1} \log(2)}{1 + \log(2) + \frac{2r+1}{a+1} \log(r+1)}, \]

for all $1 \leq r < a/2$. 


For each odd integer \( a > 1 \), there exists an \( r \) (that can be made explicit) such that the inequality (3) reduces to \( \delta(a) \geq \log(a)/3 \).

The proof of this property can be adapted to show that \( \delta(169) > 2 \), which means that there exists an integer \( j \), \( 5 \leq j \leq 169 \), such that \( 1, \zeta(3), \zeta(j) \) are linearly independent over \( \mathbb{Q} \).

4. At Least One Number Amongst \( \zeta(5), \zeta(7), \ldots, \zeta(21) \) Is Irrational

The linear independence of \( 1, \zeta(3), \zeta(j) \) for some \( j \leq 169 \) implies the irrationality of \( \zeta(j) \), but is stronger. The bound 169 is improved in this section by only seeking the irrationality.

**Theorem 4.** There exists an integer \( j \), \( 5 \leq j \leq 21 \), such that \( \zeta(j) \) is irrational.

The proof of this theorem follows the same directions as the two previous ones. First an adequate generating function \( S_n(z) \) is considered, that gives a linear equation implying the zeta function on odd integers when specialized. The coefficients of this equation are studied, and their denominator bounded; a saddle-point method gives asymptotic results on \( S_n(1) \). These lemmas, combined with the Nesterenko criterion finally give the result.

The generating function \( S_n(z) \) is

\[
S_n(z) = n^{a-6} \sum_{k=1}^{\infty} \frac{1}{2} \frac{d^2}{dk^2} \left( \frac{k-n}{2} \right) \frac{(k-n)^3(k+n+1)^3}{(k)^a_{n+1}} z^{-k},
\]

where \( a \) is an integer \( \geq 6 \). This sum is convergent when \( |z| \geq 1 \). This sum is expanded in simple elements, and then specialized at \( z = 1 \) to give a relation between values of \( \zeta \) on odd integers, \( \zeta(3) \) excluded, namely

\[
(4) \quad S_n(1) = P_{0,n}(1) + \sum_{j=2}^{a/2} j(2j-1)P_{2j-1,n}(1)\zeta(2j+1).
\]

The coefficients \( P_{l,n} \) satisfy \( 2a^{n+2}P_{0,n}(1) \in \mathbb{Z} \) and \( 2a^{n-1}P_{l,n}(1) \in \mathbb{Z} \) for \( 1 \leq l \leq a \).

The next step of the proof is to get an asymptotic result on \( S_n(1) \), using a saddle-point method. We do not know of any integral representation similar to (2) for \( S_n(1) \), but we can express \( S_n(1) \) as the real part of a complex integral. First we introduce \( R_n(k) \),

\[
R_n(k) = n^{a-6} \left( k + \frac{n}{2} \right) \frac{(k-n)^3(k+n+1)^3}{(k)^a_{n+1}}.
\]

So that \( S_n(z) = \sum_{k=1}^{\infty} \frac{1}{2} \frac{d^2}{dk^2} R_n(k) z^{-k} \). We also define

\[
J_n(u) = \frac{n}{2i\pi} \int_L R_n(nz) \left( \frac{\pi}{\sin(n\pi z)} \right)^3 e^{nuz} dz,
\]

where \( L \) is a vertical line from \( i\infty \) to \( -i\infty \) with a real part between 0 and 1. With those notations, the property \( S_n(1) = \Re(J_n(i\pi)) \) holds.

The quantity \( J_n(i\pi) \) is rewritten in terms of the \( \Gamma \) function, using the complement formula \( \Gamma(t)\Gamma(1-t) = \pi/\sin(\pi t) \), and is then approximated using the Stirling formula. This gives

\[
J_n(i\pi) = \left( i(-1)^{a+1}(2\pi)^{a/2-1} n^{-a/2} \int_L g(z) e^{nu(z)} dz \right) (1 + O(1/n)),
\]

where \( g(z) = (z+1/2) \frac{\sqrt{1-z^2}}{\sqrt{1+z^2}} \) and \( u(z) = (a+3)z \log(z) - (a+3)(z+1) \log(z+1) + 3(1-z) \log(1-z) + 3(z+2) \log(z+2) + i\pi z \). The variable \( a \) is now specialized to 20 in order to have a relation between \( \zeta(5), \ldots, \zeta(21) \). The saddle-point method, see [2, pp. 279–285], now
applies to the point $z_0$, the only root of $w'(z) = 0$ such that $0 < \Re(z) < 1$. The numerical value of $z_0$ is $0.992 - 0.012i$. The estimation of $J_n(i\pi)$ obtained is

$$J_n(i\pi) = u_n r(-1)^n n^{-8} e^{nw(z_0) + i\beta},$$

with $r$ and $\beta$ real constants and $u_n$ a sequence of complex numbers converging to 1. We define $v_0 = \Im(w(z_0))$. The real part of this expression is

$$r(-1)^n n^{-8} \Re(e^{nw(z_0) + i\beta}) = (\Re(u_n) \cos(nv_0 + \beta) - \Im(u_n) \sin(nv_0 + \beta)).$$

Since $v_0 \approx 3.104$ is not a multiple of $\pi$, there exists an increasing sequence $\phi(n)$ such that $\cos(\phi(n)v_0 + \beta)$ tends to a limit $l \neq 0$. As a direct consequence

$$\lim_{n \to \infty} \Re J_{\phi(n)}(i\pi) = K(-1)^{\phi(n)+1} \phi(n)^{-8} e^{\Re(\phi(n)w(z_0))},$$

where $K$ is a constant. So $\lim_{n \to \infty} |S_{\phi(n)}(1)|^{1/\phi(n)} = e^{\Re(w(z_0))}.$

This result, combined with Equation (4) proves Theorem 4 as follows. Equation (4) tells that $l_n = 2d_{n}^{22} S_n(1)$ is a linear combination of $\zeta(5), \ldots, \zeta(21)$ with integer coefficients. The paragraph above shows that $l_n$ satisfies $\lim_{n \to \infty} |l_{\phi(n)}|^{1/\phi(n)} \in (0, 1)$. So one of the values $\zeta(5), \ldots, \zeta(21)$ is irrational.

This result has been refined by Zudilin [6], who proved that at least one of the four numbers $\zeta(5), \zeta(7), \zeta(9), \zeta(11)$ is irrational, by using a general hypergeometric construction of linear forms in odd zeta values.

**Bibliography**