Formulas of Freitag&Gould, Knuth, Sloane, Fasler; *et al.* for NJASloane OEIS A002024; (Sunday, 10/20/2019; 20201111 SWWLiu 20OCT2019)

Closed-form expressions for $a(n)$, the nth term of OEIS A002024:

Neil J. A. Sloane OEIS A002024:
$\{1, 2, 2, 3, 3, 3, 4, 4, 4, 4, 5, 5, \ldots\}$

Knuth(1968)_1 (§0.01)

$$\boxed{a(n) = \lfloor(\sqrt{8n+1}+1)/2\rfloor = \left\lfloor\left(\sqrt{2n+\tfrac{1}{4}}\right)+\tfrac{1}{2}\right\rfloor}$$

(Donald E. Knuth, 1968, 1973;
Neil J. A. Sloane, 2009),

Knuth(1968)_2 (§0.02)

$$\boxed{a(n) = \lceil(\sqrt{8n+1}-1)/2\rceil = \left\lceil\left(\sqrt{2n+\tfrac{1}{4}}\right)-\tfrac{1}{2}\right\rceil}$$

(Donald E. Knuth, 1968, 1973);

Knuth(1968)_3 (§0.03)

$$\boxed{a(n) = \lfloor\sqrt{2n}+\tfrac{1}{2}\rfloor}$$

(Donald E. Knuth, 1968, 1973;
Pontus von Brömssen, 2018),

Knuth(1968)_4 (§0.04)

$$\boxed{a(n) = \lceil\sqrt{2n}-\tfrac{1}{2}\rceil}$$

(Donald E. Knuth, 1968, 1973;
Branco Ćurgas 2009);

Hasler(2014) (§0.05)

$$\boxed{a(n) = \lfloor(\lfloor\sqrt{8n}\rfloor+1)/2\rfloor}$$

(Maximilian F. Hasler, 2014)

SWWLiu(2019)_1 (§0.06)

$$\boxed{a(n) = \lfloor(\sqrt{8n-1}+1)/2\rfloor}$$

(Stanley Wu-Wei Liu, 2019),

SWWLiu(2019)_2 (§0.07)

$$\boxed{a(n) = \lceil(\sqrt{8n-1}-1)/2\rceil}$$

(Stanley Wu-Wei Liu, 2019);

SWWLiu(2019)_3 (§0.08)

$$\boxed{a(n) = \lfloor(\sqrt{8n-3}+1)/2\rfloor}$$

(Stanley Wu-Wei Liu, 2019),

SWWLiu(2019)_4 (§0.09)

$$\boxed{a(n) = \lceil(\sqrt{8n-3}-1)/2\rceil}$$

(Stanley Wu-Wei Liu, 2019);

SWWLiu(2019)_5 (§0.10)

$$\boxed{a(n) = \lfloor(\sqrt{8n-5}+1)/2\rfloor}$$

(Stanley Wu-Wei Liu, 2019),

SWWLiu(2019)_6 (§0.11)

$$\boxed{a(n) = \lceil(\sqrt{8n-5}-1)/2\rceil}$$

(Stanley Wu-Wei Liu, 2019);

Knuth(1968)_5 (§0.12)

$$\boxed{a(n) = \lfloor(\sqrt{8n-7}+1)/2\rfloor = \left\lfloor\left(\sqrt{2n-\tfrac{7}{4}}\right)+\tfrac{1}{2}\right\rfloor}$$

(Donald E. Knuth, 1968, 1973;
Kenneth Hardy & Kenneth S. Williams, 1985, 1997, 2013;
Néstor Jofré, 2017),

Knuth(1968)_6 (§0.13)

$$\boxed{a(n) = \lceil(\sqrt{8n-7}-1)/2\rceil = \left\lceil\left(\sqrt{2n-\tfrac{7}{4}}\right)-\tfrac{1}{2}\right\rceil}$$

(Donald E. Knuth, 1968, 1973);

Freitag&Gould(1965) (§0.14)

$$\boxed{a(n) = \lfloor(\lfloor\sqrt{8n-7}\rfloor+1)/2\rfloor}$$

(Samuel Barnard and James Mark Child, 1939;
Herta Taussig Freitag, 1964, Math. Mag. ,
Henry Wadsworth Gould, 1965, MM$\underline{\underline{38}}$, #571)

Fourteen closed-form expressions for $a(n)$, the nth term of OEIS A002024 :
$\{1, 2, 2, 3, 3, 3, 4, 4, 4, 4, 5, 5, \ldots\}$

[This document was sent on October 22, 2019, to N. J. A. Sloane by Stanley Wu-Wei Liu (SWWLiu@gmail.com). The author is currently retired, and formerly was on the editorial staff of Physical Review Letters, Physical Review A, Physical Review B, and the American Institute of Physics journal Chinese Physics. He is a Life Member of the American Mathematical Society and the American Physical Society.]

0.1 Knuth(1968)_1

A closed-form expression for $a(n)$, the nth term of OEIS A002024 : $\{1, 2, 2, 3, 3, 3, 4, 4, 4, 4, 5, 5, \ldots\}$

Knuth(1968)_1

$$\boxed{a(n) = \lfloor(\sqrt{8n+1}+1)/2\rfloor = \left\lfloor\left(\sqrt{2n+\tfrac{1}{4}}\right)+\tfrac{1}{2}\right\rfloor}$$

(Donald E. Knuth, 1968, 1973;
Neil J. A. Sloane, 2009),

It is noted that for the nth term, $a(n)$, we have, when $a(n) = k$, $(k-1)k/2 < n \leq k(k+1)/2$, and we get, by completion of squares, that, for integer n,

$\frac{1}{2}k^2 - \frac{1}{2}k < n \leq \frac{1}{2}k^2 + \frac{1}{2}k$,

$4k^2 - 4k < 8n \leq 4k^2 + 4k$,

$(4k^2 - 4k + 1) - 1 < 8n \leq (4k^2 + 4k + 1) - 1$,

$(2k-1)^2 < (8n+1) \leq (2k+1)^2$,

$2k - 1 < \sqrt{8n+1} \leq 2k + 1$,

$k < (\sqrt{8n+1}+1)/2 \leq k + 1$,

$k = \lfloor(\sqrt{8n+1}+1)/2\rfloor = a(n)$, for integer k; so

$\boxed{a(n) = \lfloor(\sqrt{8n+1}+1)/2\rfloor}$ (Donald E. Knuth).

0.2 Knuth(1968)_2

A closed-form expression for $a(n)$, the nth term of OEIS A002024 : $\{1, 2, 2, 3, 3, 3, 4, 4, 4, 4, 5, 5, \ldots\}$

Knuth(1968)_2

$$\boxed{a(n) = \lceil(\sqrt{8n+1}-1)/2\rceil = \left\lceil\left(\sqrt{2n+\tfrac{1}{4}}\right)-\tfrac{1}{2}\right\rceil}$$

(Donald E. Knuth, 1968, 1973);

As noted in *Concrete Mathematics*, 2nd Edition (1989, 1994), by Ronald L. Graham, Donald E. Knuth, and Oren Patashnik (§3.1, Eqs. 3.2 and 3.6, pp. 68–69), for real number x not an integer,

$$(\lceil x\rceil = \lfloor x\rfloor + 1) \iff (x \notin \mathbb{Z}),$$

which, when applied to the preceding formula,

$$a(n) = \lfloor(\sqrt{8n+1}+1)/2\rfloor,$$

yields

$\boxed{a(n) = \lceil(\sqrt{8n+1}-1)/2\rceil}$ (Donald E. Knuth).

0.3 Knuth(1968)_3

A closed-form expression for $a(n)$, the nth term of OEIS A002024 : $\{1, 2, 2, 3, 3, 3, 4, 4, 4, 4, 5, 5, \ldots\}$

Knuth(1968)_3

$$\boxed{a(n) = \lfloor\sqrt{2n}+\tfrac{1}{2}\rfloor}$$

(Donald E. Knuth, 1968, 1973;
Pontus von Brömssen, 2018),

Knuth notes that the nth term, $a(n)$, obeys $a(n) = k$ when $(k-1)k/2 < n \leq k(k+1)/2$. Since n is an integer, this is equivalent to $\frac{1}{2}(k-1)k + \frac{1}{4} < n < \frac{1}{2}k(k+1) + \frac{1}{4}$, or $k - \frac{1}{2} < \sqrt{2n} < k + \frac{1}{2}$; hence the formula $k = \boxed{a(n) = \lfloor\sqrt{2n}+\frac{1}{2}\rfloor}$ (Donald E. Knuth).

This approach to solution may be called "**Knuth's TNS-COSA**" (*Triangular-Number Sandwich, Completion of Squares Approach*), and it is used in nearly all known solutions for a closed-form expression of the nth term in Sloane's OEIS A002024 integer sequence.

0.3.1 Knuth(1968)_3 ; SWWLiu Note 1

Donald E. Knuth (*The Art of Computer Programming*, Volume 1: Fundamental Algorithms, 1st Edition, 1968, 2nd Edition, 1973, p. 478, in his Solution to Problem 41 [M23] of §1.2.4, p. 43), in addition to the proof of the formula given above, also asserted (skipping the proof, apparently for space economy) that the following two formulas for the nth term are correct:

$$a(n) = \lceil (-1 + \sqrt{8n+1})/2 \rceil,$$

$$a(n) = \lfloor (1 + \sqrt{8n-7})/2 \rfloor$$

[the latter of which had been stated, in a variant form (with nested floor functions), by Samuel Barnard and James Mark Child (1939, p. 271) as Problem 91 of their *Advanced Algebra* book, according to Henry Wadsworth Gould (1965, p. 186) of *West Virginia University* in his "Solution to Problem 571" (proposed by Herta Taussig Freitag of *Hollins College, Virginia*) in *Mathematics Magazine* 38, 185–187 (1965)].

0.3.2 Knuth(1968)_3 ; SWWLiu Note 2

In Chapter 3 of *Concrete Mathematics: A Foundation for Computer Science*, 2nd Edition (1989, 1994), by Ronald L. Graham, Donald E. Knuth, and Oren Patashnik, the same problem (Problem 41 [M23] of Knuth) occurs in Exercise 3.23 (on page 97): "Show that the nth element of the sequence $1, 2, 2, 3, 3, 3, 4, 4, 4, 4, 5, 5, 5, 5, 5, \ldots$ is $\lfloor \sqrt{2n} + \frac{1}{2} \rfloor$. (The sequence contains exactly m occurrences of m.)" The Answer to Exercise 3.23 is given (on page 508): "$X_n = m \iff \frac{1}{2}m(m-1) < m \le \frac{1}{2}m(m+1) \iff m^2 - m + \frac{1}{4} < 2n < m^2 + m + \frac{1}{4} \iff m - \frac{1}{2} < \sqrt{2n} < m + \frac{1}{2}$."

0.4 Knuth(1968)_4

A closed-form expression for $a(n)$, the nth term of OEIS A002024 :
$\{1, 2, 2, 3, 3, 3, 4, 4, 4, 4, 5, 5, \ldots\}$

Knuth(1968)_4

$$\boxed{a(n) = \lceil \sqrt{2n} - \tfrac{1}{2} \rceil}$$

(Donald E. Knuth, 1968, 1973;
Branco Ćurgas 2009);

As noted in *Concrete Mathematics*, 2nd Edition (1989, 1994), by Ronald L. Graham, Donald E. Knuth, and Oren Patashnik (§3.1, Eqs. 3.2 and 3.6, pp. 68–69), for real number x not an integer,

$$(\lceil x \rceil = \lfloor x \rfloor + 1) \iff (x \notin \mathbb{Z}),$$

which, when applied to the preceding formula,

$$a(n) = \lfloor \sqrt{2n} + \tfrac{1}{2} \rfloor,$$

yields

$$\boxed{a(n) = \lceil \sqrt{2n} - \tfrac{1}{2} \rceil} \text{ (Donald E. Knuth).}$$

0.5 Hasler(2014)

A closed-form expression for $a(n)$, the nth term of OEIS A002024 :
$\{1, 2, 2, 3, 3, 3, 4, 4, 4, 4, 5, 5, \ldots\}$

Hasler(2014)

$$\boxed{a(n) = \lfloor (\lfloor \sqrt{8n} \rfloor + 1)/2 \rfloor}$$

(Maximilian F. Hasler, 2014)

Hasler notes that, with $\lfloor x \rfloor \le x < \lfloor x \rfloor + 1$, if we let $a(x) = \lfloor x \rfloor$ and $x = \sqrt{2n} + \frac{1}{2} = (\sqrt{8n} + 1)/2$ and substitute them into the inequality, we get

$$a(x) \le (\sqrt{8n} + 1)/2 < a(x) + 1,$$

$$2a(x) - 1 \le \sqrt{8n} < 2a(x) + 1,$$

Since $a(x)$ is an integer, the left-hand side of the preceding inequality is an integer and the fractional part of $\sqrt{8n}$ does not matter for the value of $a(n)$: any value between $\lfloor \sqrt{8n} \rfloor$ and strictly less than $\lceil \sqrt{8n} \rceil$ will give the same $a(n)$. Hence,

$$\boxed{a(n) = \lfloor (\lfloor \sqrt{8n} \rfloor + 1)/2 \rfloor} \text{ (Maximilian F. Hasler).}$$

0.5.1 Hasler(2014) ; SWWLiu Note 1

In *Concrete Mathematics: A Foundation for Computer Science*, 2nd Edition (1989, 1994), by Ronald L. Graham, Donald E. Knuth, and Oren Patashnik (§3.2, Eqs. 3.10 and 3.11, pp. 70–74), for any continuous, monotonically increasing function $f(x)$ on an interval of the real numbers, with the property that

$$(f(x) = \text{integer}) \implies (x = \text{integer}),$$

it is shown that

$$\lfloor f(x) \rfloor = \lfloor f(\lfloor x \rfloor) \rfloor \text{ and } \lceil f(x) \rceil = \lceil f(\lceil x \rceil) \rceil,$$

whenever $f(x), f(\lfloor x \rfloor)$, and $f(\lceil x \rceil)$ are defined.

In particular,

$$\lfloor \tfrac{x+m}{n} \rfloor = \lfloor \tfrac{\lfloor x \rfloor + m}{n} \rfloor \text{ and } \lceil \tfrac{x+m}{n} \rceil = \lceil \tfrac{\lceil x \rceil + m}{n} \rceil,$$

if m and n are integers and the denominator n is positive.

Therefore, application of Eqs. 3.10 and 3.11 of Graham, Knuth, and Patashnik (1989, 1994) to

$$a(n) = \lfloor (\sqrt{8n} + 1)/2 \rfloor = \lfloor \sqrt{2n} + \tfrac{1}{2} \rfloor$$
(Donald E. Knuth)

yields

$$\boxed{a(n) = \lfloor (\lfloor \sqrt{8n} \rfloor + 1)/2 \rfloor}$$ (Maximilian F. Hasler).

0.6 SWWLiu(2019)_1

A closed-form expression for $a(n)$, the nth term of OEIS A002024 : $\{1, 2, 2, 3, 3, 3, 4, 4, 4, 4, 5, 5, \ldots\}$

SWWLiu(2019)_1

$$\boxed{a(n) = \lfloor (\sqrt{8n-1} + 1)/2 \rfloor}$$
(Stanley Wu-Wei Liu, 2019).

It is noted that for the nth term, $a(n)$, we have, when $a(n) = k$, $(k-1)k/2 < n \le k(k+1)/2$, and we get, by completion of squares, that, for integer n,

$$\tfrac{1}{2}k^2 - \tfrac{1}{2}k + \tfrac{1}{4} < n < \tfrac{1}{2}k^2 + \tfrac{1}{2}k + \tfrac{1}{4},$$

$$4k^2 - 4k + 2 < 8n < 4k^2 + 4k + 2,$$

$$4k^2 - 4k + 1 < 8n - 1 < 4k^2 + 4k + 1,$$

$$(2k-1)^2 < (8n-1) < (2k+1)^2,$$

$$2k - 1 < \sqrt{8n-1} < 2k + 1,$$

$$k < (\sqrt{8n-1} + 1)/2 < k + 1,$$

$k = \lfloor (\sqrt{8n-1} + 1)/2 \rfloor = a(n)$, for integer k; so

$$\boxed{a(n) = \lfloor (\sqrt{8n-1} + 1)/2 \rfloor}$$ (Stanley Wu-Wei Liu).

0.7 SWWLiu(2019)_2

A closed-form expression for $a(n)$, the nth term of OEIS A002024 : $\{1, 2, 2, 3, 3, 3, 4, 4, 4, 4, 5, 5, \ldots\}$

SWWLiu(2019)_2

$$\boxed{a(n) = \lceil (\sqrt{8n-1} - 1)/2 \rceil}$$
(Stanley Wu-Wei Liu, 2019),

As noted in *Concrete Mathematics*, 2nd Edition (1989, 1994), by Ronald L. Graham, Donald E. Knuth, and Oren Patashnik (§3.1, Eqs. 3.2 and 3.6, pp. 68–69), for real number x not an integer,

$$(\lceil x \rceil = \lfloor x \rfloor + 1) \iff (x \notin \mathbb{Z}),$$

which, when applied to the preceding formula,

$$a(n) = \lfloor (\sqrt{8n-1} + 1)/2 \rfloor,$$

yields

$\boxed{a(n) = \lceil (\sqrt{8n-1} - 1)/2 \rceil}$ (Stanley Wu-Wei Liu).

0.8 SWWLiu(2019)_3

A closed-form expression for $a(n)$, the nth term of OEIS A002024 :
$\{1, 2, 2, 3, 3, 3, 4, 4, 4, 4, 5, 5, \ldots\}$

SWWLiu(2019)_3
$\boxed{a(n) = \lfloor (\sqrt{8n-3} + 1)/2 \rfloor}$
(Stanley Wu-Wei Liu, 2019).

It is noted that for the nth term, $a(n)$, we have, when $a(n) = k$, $(k-1)k/2 < n \le k(k+1)/2$, and we get, by completion of squares, that, for integer n,

$\frac{1}{2}k^2 - \frac{1}{2}k + \frac{1}{2} < n < \frac{1}{2}k^2 + \frac{1}{2}k + \frac{1}{2}$,

$4k^2 - 4k + 4 < 8n < 4k^2 + 4k + 4$,

$4k^2 - 4k + 1 < 8n - 3 < 4k^2 + 4k + 1$,

$(2k-1)^2 < (8n-3) < (2k+1)^2$,

$2k - 1 < \sqrt{8n-3} < 2k + 1$,

$k < (\sqrt{8n-3} + 1)/2 < k + 1$,

$k = \lfloor (\sqrt{8n-3} + 1)/2 \rfloor = a(n)$, for integer k; so

$\boxed{a(n) = \lfloor (\sqrt{8n-3} + 1)/2 \rfloor}$ (Stanley Wu-Wei Liu).

0.9 SWWLiu(2019)_4

A closed-form expression for $a(n)$, the nth term of OEIS A002024 :
$\{1, 2, 2, 3, 3, 3, 4, 4, 4, 4, 5, 5, \ldots\}$

SWWLiu(2019)_4
$\boxed{a(n) = \lceil (\sqrt{8n-3} - 1)/2 \rceil}$
(Stanley Wu-Wei Liu, 2019).

As noted in *Concrete Mathematics*, 2nd Edition (1989, 1994), by Ronald L. Graham, Donald E. Knuth, and Oren Patashnik (§3.1, Eqs. 3.2 and 3.6, pp. 68–69), for real number x not an integer,

$$(\lceil x \rceil = \lfloor x \rfloor + 1) \iff (x \notin \mathbb{Z}),$$

which, when applied to the preceding formula,

$$a(n) = \lfloor (\sqrt{8n-3} + 1)/2 \rfloor,$$

yields

$\boxed{a(n) = \lceil (\sqrt{8n-3} - 1)/2 \rceil}$ (Stanley Wu-Wei Liu).

0.10 SWWLiu(2019)_5

A closed-form expression for $a(n)$, the nth term of OEIS A002024 :
$\{1, 2, 2, 3, 3, 3, 4, 4, 4, 4, 5, 5, \ldots\}$

SWWLiu(2019)_5
$\boxed{a(n) = \lfloor (\sqrt{8n-5} + 1)/2 \rfloor}$
(Stanley Wu-Wei Liu, 2019).

It is noted that for the nth term, $a(n)$, we have, when $a(n) = k$, $(k-1)k/2 < n \le k(k+1)/2$, and we get, by completion of squares, that, for integer n,

$\frac{1}{2}k^2 - \frac{1}{2}k + \frac{3}{4} < n < \frac{1}{2}k^2 + \frac{1}{2}k + \frac{3}{4}$,

$4k^2 - 4k + 6 < 8n < 4k^2 + 4k + 6$,

$4k^2 - 4k + 1 < 8n - 5 < 4k^2 + 4k + 1$,

$(2k-1)^2 < (8n-5) < (2k+1)^2$,

$2k - 1 < \sqrt{8n-5} < 2k + 1$,

$k < (\sqrt{8n-5} + 1)/2 < k + 1$,

$k = \lfloor (\sqrt{8n-5} + 1)/2 \rfloor = a(n)$, for integer k; so

$\boxed{a(n) = \lfloor(\sqrt{8n-5}+1)/2\rfloor}$ (Stanley Wu-Wei Liu).

0.11 SWWLiu(2019)_6

A closed-form expression for $a(n)$, the nth term of OEIS A002024 :
$\{1, 2, 2, 3, 3, 3, 4, 4, 4, 4, 5, 5, \ldots\}$

SWWLiu(2019)_6
$\boxed{a(n) = \lceil(\sqrt{8n-5}-1)/2\rceil}$
(Stanley Wu-Wei Liu, 2019).

As noted in *Concrete Mathematics*, 2nd Edition (1989, 1994), by Ronald L. Graham, Donald E. Knuth, and Oren Patashnik (§3.1, Eqs. 3.2 and 3.6, pp. 68–69), for real number x not an integer,

$$(\lceil x\rceil = \lfloor x\rfloor + 1) \Longleftrightarrow (x \notin \mathbb{Z}),$$

which, when applied to the preceding formula,

$$a(n) = \lfloor(\sqrt{8n-5}+1)/2\rfloor,$$

yields

$\boxed{a(n) = \lceil(\sqrt{8n-5}-1)/2\rceil}$ (Stanley Wu-Wei Liu).

0.12 Knuth(1968)_5

A closed-form expression for $a(n)$, the nth term of OEIS A002024 :
$\{1, 2, 2, 3, 3, 3, 4, 4, 4, 4, 5, 5, \ldots\}$

Knuth(1968)_5
$\boxed{a(n) = \lfloor(\sqrt{8n-7}+1)/2\rfloor = \left\lfloor\left(\sqrt{2n-\frac{7}{4}}\right)+\frac{1}{2}\right\rfloor}$
(Donald E. Knuth, 1968, 1973;
Kenneth Hardy & Kenneth S. Williams, 1985, 1997, 2013;
Néstor Jofré, 2017),

It is noted that for the nth term, $a(n)$, we have, when $a(n) = k$, $(k-1)k/2 < n \le k(k+1)/2$, and we get, by completion of squares, that, for integer n,

$$\tfrac{1}{2}k^2 - \tfrac{1}{2}k + \tfrac{7}{8} < n < \tfrac{1}{2}k^2 + \tfrac{1}{2}k + \tfrac{7}{8},$$

$$4k^2 - 4k + 7 < 8n < 4k^2 + 4k + 7,$$

$$4k^2 - 4k < 8n - 7 < 4k^2 + 4k,$$

$$(4k^2 - 4k + 1) - 1 < 8n - 7 < (4k^2 + 4k + 1) - 1,$$

$$(2k-1)^2 \le (8n-7) < (2k+1)^2,$$

$$2k - 1 \le \sqrt{8n-7} < 2k + 1,$$

$$k \le (\sqrt{8n-7}+1)/2 < k + 1,$$

$k = \lfloor(\sqrt{8n-7}+1)/2\rfloor = a(n)$, for integer k; so

$\boxed{a(n) = \lfloor(\sqrt{8n-7}+1)/2\rfloor}$ (Donald E. Knuth).

0.12.1 Knuth(1968)_5 ; SWWLiu Note 1

In *The Green Book of Mathematical Problems* (1985, 1997, 2013), by Kenneth Hardy and Kenneth S. Williams, the same problem (Problem 91 [M23] of Knuth) occurs as Problem 14 (with Solution to Problem 14 on pages 59–60); in our notation, the solution of Hardy and Williams (1985) may be reproduced as follows:

Let $a(n)$ be the nth term of the sequence $1, 2, 2, 3, 3, 3, 4, 4, 4, 4, 5, \ldots$. The integer k first occurs when $n = 1+2+3+\cdots+(k-1)+1 = \frac{(k-1)k}{2}+1$. Hence $a(n) = k$ for $n = \frac{(k-1)k}{2} + 1 + m$, with $m = 0, 1, 2, \ldots, k-1$, from which we obtain $0 \le n - \frac{(k-1)k}{2} - 1 \le k - 1$ and so $\frac{k^2-k+2}{2} \le n \le \frac{k^2+k}{2}$; $(2k-1)^2 + 7 \le 8n \le (2k+1)^2 - 1$; $(2k-1)^2 \le 8n - 7 \le (2k+1)^2 - 8 < (2k+1)^2$; $2k-1 \le \sqrt{8n-7} < 2k+1$; $k \le (\sqrt{8n-7}+1)/2 < k+1$; so that $k = \boxed{a(n) = \lfloor(\sqrt{8n-7}+1)/2\rfloor}$.

0.13 Knuth(1968)_6

A closed-form expression for $a(n)$, the nth term of OEIS A002024 :

$\{1, 2, 2, 3, 3, 3, 4, 4, 4, 4, 5, 5, \ldots\}$

Knuth(1968)_6

$$\boxed{a(n) = \lceil(\sqrt{8n-7}-1)/2\rceil = \left\lceil\left(\sqrt{2n-\tfrac{7}{4}}\right)-\tfrac{1}{2}\right\rceil}$$

(Donald E. Knuth, 1968, 1973);

As noted in *Concrete Mathematics*, 2nd Edition (1989, 1994), by Ronald L. Graham, Donald E. Knuth, and Oren Patashnik (§3.1, Eqs. 3.2 and 3.6, pp. 68–69), for real number x not an integer,

$$(\lceil x\rceil = \lfloor x\rfloor + 1) \iff (x \notin \mathbb{Z}),$$

which, when applied to the preceding formula,

$$a(n) = \lfloor(\sqrt{8n-7}+1)/2\rfloor,$$

yields

$$\boxed{a(n) = \lceil(\sqrt{8n-7}-1)/2\rceil} \text{ (Donald E. Knuth).}$$

0.14 Freitag&Gould(1965)

A closed-form expression for $a(n)$, the nth term of OEIS A002024 :
$\{1, 2, 2, 3, 3, 3, 4, 4, 4, 4, 5, 5, \ldots\}$

Freitag&Gould(1965)

$$\boxed{a(n) = \lfloor(\lfloor\sqrt{8n-7}\rfloor+1)/2\rfloor}$$

(Samuel Barnard and James Mark Child, 1939;
Herta Taussig Freitag, 1964, Math. Mag. ,
Henry Wadsworth Gould, 1965, MM38, #571)

In *Concrete Mathematics: A Foundation for Computer Science*, 2nd Edition (1989, 1994), by Ronald L. Graham, Donald E. Knuth, and Oren Patashnik (§3.2, Eqs. 3.10 and 3.11, pp. 70–74), for any continuous, monotonically increasing function $f(x)$ on an interval of the real numbers, with the property that

$$(f(x) = \text{integer}) \implies (x = \text{integer}),$$

it is shown that

$$\lfloor f(x)\rfloor = \lfloor f(\lfloor x\rfloor)\rfloor \text{ and } \lceil f(x)\rceil = \lceil f(\lceil x\rceil)\rceil,$$

whenever $f(x), f(\lfloor x\rfloor)$, and $f(\lceil x\rceil)$ are defined.

In particular,

$$\lfloor \tfrac{x+m}{n}\rfloor = \lfloor \tfrac{\lfloor x\rfloor+m}{n}\rfloor \text{ and } \lceil \tfrac{x+m}{n}\rceil = \lceil \tfrac{\lceil x\rceil+m}{n}\rceil,$$

if m and n are integers and the denominator n is positive.

Therefore, application of Eqs. 3.10 and 3.11 of Graham, Knuth, and Patashnik (1989, 1994) to

$$a(n) = \lfloor(\sqrt{8n-7}+1)/2\rfloor \text{ (Donald E. Knuth)}$$

yields

$$\boxed{a(n) = \lfloor(\lfloor\sqrt{8n-7}\rfloor+1)/2\rfloor}$$

(Herta Taussig Freitag and Henry W. Gould).

This formula was first given (on page 186) by Henry Wadsworth Gould of *West Virginia University* in his "Solution to Problem 571" (proposed by Herta Taussig Freitag of *Hollins College, Virginia*), in *Mathematics Magazine* 38, 185–187 (1965); according to Gould, this variant form (with nested floor functions) had been stated by Samuel Barnard and James Mark Child (1939, 1953) as Problem 91 (on page 271) of their *Advanced Algebra* book.