Majority Decision Functions of up to Six Variables

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1. Introduction. Recently logical elements based essentially on the majority decision principle have been widely used in electronic computers. Among these elements are parametrons, magnetic cores, transistor-resistor logic, et cetera.

The logical behavior of such elements can be expressed by a model called a "majority decision element" with n Boolean inputs and one Boolean output, whose operation can be described in the form of a logical function called a "majority decision function".

This paper defines the canonical representative of each equivalence class in the classification of the majority decision functions by complementing and permuting variables and by complementing the output. Also, a method is proposed to obtain all the representatives with their optimum structures, and a table of the representatives of the majority decision functions of up to six variables is provided.

The reader should be familiar with the content of a previous paper by the authors, included as reference [1].

2. Majority Decision Functions. A "majority decision element" of n variables is a logical element with n Boolean inputs, $x_1, x_2, \cdots, x_n$ and one Boolean output. The output value of the element is

$$(1)^* \qquad \begin{aligned} &\text{one for} \quad \sum_{i=1}^{n} w_i x_i \geqq T \\ &\text{zero for} \quad \sum_{i=1}^{n} w_i x_i \leqq T - 1 \end{aligned}$$

where w_i is a prescribed constant real number called a "coupling weight" associated with the input x_i and T is also a prescribed constant real number called a "threshold."

In the case of parametrons or magnetic cores, the coupling weight w_i corresponds to the number of turns of the winding of the input x_i. The threshold T is related to the number of turns w_c for the constant input by the relation,

$$(2) \qquad w_c = \sum_{i=1}^{n} w_i + 1 - 2T$$

where $w_c \geq 0$ means the constant of one is coupled to the element and $w_c < 0$ means the constant of zero.

A set of $(n + 1)$ real numbers $(w_1, w_2, \cdots, w_n; T)$, which specifies the behavior of a majority decision element, will be called a "structure" of the element.

A logical function represented by a single majority decision element will be called a "majority decision function."

Received September 22, 1961.

* The term -1 on the right hand side is introduced as a normalizing factor of w_i's and T.

For example, a majority decision element with the structure (2, 1, 1; 2) represents a function $x_1 + x_2x_3$; hence, this function is a majority decision function. In contrast, the function $x_1x_2 + x_3x_4$ is not a majority decision function since it can not be realized by any single majority decision element.

3. Classification of Majority Decision Functions. Logical functions obtained from a given logical function f by the following operations are defined as equivalent functions with f:

(1) Complementation of one or more input variables,
(2) Permutation among input variables,
(3) Complementation of f.

It is a well known fact that the logical functions can be classified into equivalent classes by this equivalent relation. Once a structure of a majority decision function is given, its equivalent functions can be easily realized in the same element by complementing and/or permuting input variables and/or by complementing the output. Thus, it is not necessary to determine the whole of the majority decision functions; it is sufficient to know the representatives of their equivalence classes. It should be noted that this limits the study to a much smaller number of functions.

In the case of general logical functions, it is difficult to extract systematically one representative from each equivalence class, but in the case of majority decision functions there is a way to define a canonical representative of each equivalence class from the intrinsic nature of majority decision functions.

The method of determining the canonical representative is described below. Hereafter in this section the majority decision function is assumed to have n non-vacuous variables.

Any majority decision function can be expressed by a polynomial without any complemented variable by appropriately complementing one or more variables (refer to [1], Section 3). Such a polynomial will be called a "positive polynomial." The way to complement the variables to obtain a positive polynomial from a given function is unique if complementing one variable more than once is prohibited. Hence we can restrict the possible representatives within positive polynomials. This is equivalent to the condition in which the representative should be realized by a majority decision element with positive coupling weights.

All the variables of a majority decision function can be ordered by a relation $\gtrsim$ (refer to [1], Definition 3 and Theorem 1). Therefore, it is always possible for variables to be permuted and relabelled so that $x_1 \gtrsim x_2 \gtrsim \cdots \gtrsim x_m$ holds. This permutation can be uniquely determined except in the case of arbitrary permutations among some variables such as $x_1, x_2, \cdots, x_m$ for which $x_1 \sim x_2 \sim \cdots \sim x_m$ holds. But $x_1 \sim x_2 \sim \cdots \sim x_m$ means that the given function is symmetric with respect to these variables, and therefore the function is invariant under the permutations among $x_1, x_2, \cdots, x_m$. Thus, the function for which $x_1 \gtrsim x_2 \gtrsim \cdots \gtrsim x_n$ holds is unique and can well be adopted as a possible representative. Of course, this is equivalent to the condition in which $w_1 \geqq w_2 \geqq \cdots \geqq w_n$ holds for the representative majority decision element. Note that as a conclusion from the above requirements, we have $w_1 \geqq w_2 \geqq \cdots \geqq w_n > 0$ except $w_c \leqq 0$.

Only two functions left in each class satisfy both of the conditions just described.

If we denote one of them by f, the other is the dual function f^* of f. But for a majority decision function, either $f^* \supseteq f$, or $f \supseteq f^*$ holds (refer to [1], Corollary 2). A unique representative of the equivalent class can be determined by requiring either of the two inequalities. If we adopt f such that $f \subseteq f^*$, this implies $w_c \leqq 0$.

Thus, it is shown that there is a unique canonical representative in each equivalent class of majority decision functions which satisfies the following three conditions:

Conditions I.

(1) A positive polynomial,
(2) $x_1 \gtrsim x_2 \gtrsim \cdots \gtrsim x_n$,
(3) f such that $f \subseteq f^*$.

Given a majority decision function, we can now effectively obtain the representative of the equivalent class to which the given function belongs.

4. A Method to Obtain the Totality of the Representatives of the Majority Decision Functions. From Section 5 of [1] it can be determined by linear programming whether a given function is a majority decision function or not. Therefore, it is possible, at least in principle, to obtain the totality of majority decision functions by applying the criterion to all of 2^{2^n} logical functions of n variables. It will, however, take an impractically long time to solve 2^{2^n} linear programming problems for large values of n, but the length of time to perform computation will be greatly reduced if we can confine the scope of the functions to be tested.

Accordingly, a method is developed here to obtain a set of logical functions which includes all the representatives of majority decision functions and to apply the criterion only to those functions in the set. The functions in the set will be called "candidates" of the representatives.

Any positive majority decision function can be expressed in the form of $Mx_1 + N$, where M and N are both positive majority decision functions of $(n - 1)$ variables, $x_2, x_3, \cdots, x_n$. Therefore, without loss of generality, we can restrict the candidates within such functions. This assumes that we have already obtained all the majority decision functions of $(n - 1)$ variables; hence the method described here is one of the recursive constructions of majority decision functions with respect to the number of variables.

Moreover, if we choose as the candidates those functions for which Conditions I can be defined, then the set of the candidates will certainly contain the totality of the representatives of the majority decision functions of n variables.

Then the restrictions imposed upon combinations of M and N will be examined.

Condition (1) will be trivially satisfied, for $Mx_1 + N$ is positive from its construction.

Condition (2) requires that the relation

$$x_2 \gtrsim x_3 \gtrsim \cdots \gtrsim x_n \tag{3}$$

must hold for both M and N. Moreover, in order that $x_1 \gtrsim x_2$ may hold in $Mx_1 + N$, it is necessary (Corollary 1 of Reference [1]), that

$$m_2 \supseteq n_1, \tag{4}$$

where

$$m_2 = M(0, x_3, \cdots, x_n)$$
$$n_1 = N(1, x_3, \cdots, x_n).$$

As the relation $\gtrsim$ is an ordering relation (Theorem 1 of [1]), the relation

$$x_1 \gtrsim x_2 \gtrsim \cdots \gtrsim x_n \tag{5}$$

follows from (3) and (4).

M and N are majority decision functions satisfying (3), hence the relations

$$m_1 \supseteq m_2 \quad \text{and} \quad n_1 \supseteq n_2 \tag{6}$$

where

$$m_1 = M(1, x_3, \cdots, x_n)$$
$$n_2 = N(0, x_3, \cdots, x_n)$$

hold (Corollary 1 of Reference [1]). From (4) and (6) we have

$$M \supset N. \tag{7}$$

From (3) in Conditions I, it is necessary that

$$f^* = N^*x_1 + M^*N^* \supseteq f = Mx_1 + N. \tag{8}$$

But as $M^*N^* = M^*$ from (7), (8) reduces to

$$M^* \supseteq N. \tag{9}$$

Thus, we choose as candidates those functions which satisfy the following conditions:

Conditions II

(1) Both M and N are positive majority decision functions of $(n - 1)$ variables, $x_2, x_3, \cdots, x_n$.
(2) For both M and N, $x_2 \gtrsim x_3 \gtrsim \cdots \gtrsim x_n$.
(3) $m_2 \supseteq n_1$.
(4) $M^* \supseteq N$.

By taking all the combinations of M and N which satisfy Conditions II, we can obtain the set of candidates of the representatives of majority decision functions of n variables.

M and N must satisfy (1) and (2) of Conditions II. Such functions are either canonical representatives of majority decision functions or their dual functions. Therefore, once the totality of representatives of majority decision functions of $(n - 1)$ variables are obtained, the scope within which functions M and N must be taken can be easily determined. In this way we can obtain the totality of the representatives of majority decision functions of n variables recursively.

The next problem is to examine each candidate to determine whether or not it is a majority decision function. If so, it is clearly a canonical representative of an equivalent class defined in the preceding section. The discrimination of majority decision functions from other functions can be accomplished by linear programming. The details will be found in Section 5 of [1].

5. Majority Decision Functions of up to Six Variables. Following the procedure described in Section 4, a program was written for the parametron digital computer MUSASINO-I, and all the canonical representatives of the functions of up to six variables were obtained.

The canonical representatives of up to five variables had been obtained by S. Muroga [3] at that time, using a combinatorial method. Both results agreed completely.

The canonical representatives of the functions of up to six variables are shown in Table 1. The functions are numbered according to the magnitude of $V = \sum_{i=1}^{n} w_i$, which is expected to denote the complexity of functions to some extent. Functions are expressed by denoting the variables by means of their subscripts. For instance, $12 + 13 + 23$ stands for the function $x_1x_2 + x_1x_3 + x_2x_3$.

In the same entry of the table an optimum structure of the function is shown. The optimum structure is one with a minimum number of total turns of windings, namely, a structure which minimizes $(w_1 + w_2 + \cdots + w_n + | w_c |)$ (Section 5 in [1]).

To establish the threshold T, the constant input of zero must be coupled to the element with a winding of $2T - V - 1$ turns. Dual functions can be realized by merely reversing the polarity of the constant input, that is, by coupling the constant of one to the same winding.

The numbers in this table are somewhat different from those shown in [1]. This is because f and f^* are considered to belong to the same equivalence class in this paper and that in Table 1 the numbers of functions of n (nonvacuous) variables are shown, while the numbers for up to n variables are shown in [1].

By computing the number of the members of each equivalent class, the total numbers of majority decision functions are obtained and shown in Table 2.

6. Remarks on the Results. Some remarks are added here concerning the representatives of majority decision functions of up to six variables.

First, it is remarkable that all the candidates proved to be true representatives, that is, Conditions II are sufficient for a function of up to six variables to be realized by a single majority decision element.

Second, it is interesting to note that the optimum structures $(w_1, w_2, \cdots, w_n)$ are all integer-valued in spite of the fact that the optimum structure is obtained as a solution of a system of inequalities of the form of equation (1).

A structure of a majority decision function is a solution of a system of 2^n linear inequalities (Section 5 of Reference [1]).

$$Ax \geq b \qquad A = \left\{ \begin{matrix} a_{ij}; i \to 1, 2, \cdots, 2^n \\ i \to 1, 2, \cdots, n \end{matrix} \right\} \tag{10}$$

$$x = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \\ T \end{bmatrix}$$

The third remark concerns the structure of the solution space of these inequalities. It has been noted that for a majority decision function of up to five

TABLE 1

Representative Functions of Majority Decision Functions of up to Six Variables

No.	V	w_i	T	Representative Function
				$n = 2$
1	2	11	2	12
				$n = 3$
1	3	111	2	12 + 13 + 23
2	3	111	3	123
3	4	211	3	12 + 13
				$n = 4$
1	4	1111	3	123 + 124 + 134 + 234
2	4	1111	4	1234
3	5	2111	3	12 + 13 + 14 + 234
4	5	2111	4	123 + 124 + 134
5	6	2211	4	12 + 134 + 234
6	6	2211	5	123 + 124
7	6	3111	4	12 + 13 + 14
8	7	3211	5	12 + 134
9	8	3221	5	12 + 13 + 234
				$n = 5$
1	5	11111	3	123 + 124 + 134 + 234 + 125 + 135 + 235 + 145 + 245 + 345
2	5	11111	4	1234 + 1235 + 1245 + 1345 + 2345
3	5	11111	5	12345
4	6	21111	4	123 + 124 + 134 + 125 + 135 + 145 + 2345
5	6	21111	5	1234 + 1235 + 1245 + 1345
6	7	22111	4	12 + 134 + 234 + 135 + 235 + 145 + 245
7	7	22111	5	123 + 124 + 125 + 1345 + 2345
8	7	22111	6	1234 + 1235 + 1245
9	7	31111	4	12 + 13 + 14 + 15 + 2345
10	7	31111	5	123 + 124 + 134 + 125 + 135 + 145
11	8	22211	5	123 + 124 + 134 + 234 + 125 + 135 + 235
12	8	22211	6	123 + 1245 + 1345 + 2345
13	8	22211	7	1234 + 1235
14	8	32111	5	12 + 134 + 135 + 145 + 2345
15	8	32111	6	123 + 124 + 125 + 1345
16	8	41111	5	12 + 13 + 14 + 15
17	9	32211	5	12 + 13 + 234 + 235 + 145
18	9	32211	6	123 + 124 + 134 + 125 + 135 + 2345
19	9	32211	7	123 + 1245 + 1345
20	9	33111	6	12 + 1345 + 2345
21	9	33111	7	123 + 124 + 125
22	9	42111	6	12 + 134 + 135 + 145
23	10	32221	6	123 + 124 + 134 + 234 + 125 + 135 + 145
24	10	32221	7	123 + 124 + 134 + 2345
25	10	33211	6	12 + 134 + 234 + 135 + 235
26	10	33211	8	123 + 1245
27	10	42211	6	12 + 13 + 145 + 2345
28	10	42211	7	123 + 124 + 134 + 125 + 135
29	10	43111	7	12 + 1345
30	11	33221	7	123 + 124 + 134 + 234 + 125
31	11	33221	8	123 + 124 + 1345 + 2345
32	11	43211	7	12 + 134 + 135 + 2345
33	11	52211	7	12 + 13 + 145
34	12	33222	7	123 + 124 + 134 + 234 + 125 + 135 + 235 + 145 + 245
35	12	43221	7	12 + 134 + 234 + 135 + 145
36	12	43221	8	123 + 124 + 134 + 125 + 2345
37	12	43221	9	123 + 124 + 1345
38	12	43311	7	12 + 13 + 234 + 235
39	12	52221	7	12 + 13 + 14 + 2345
40	12	53211	8	12 + 134 + 135
41	13	43321	8	123 + 124 + 125 + 135 + 134 + 234
42	13	53221	9	123 + 124 + 125 + 134
43	13	53311	8	12 + 13 + 2345
44	14	43322	8	123 + 124 + 125 + 134 + 135 + 145 + 234 + 235
45	14	53321	8	12 + 13 + 145 + 234
46	14	54221	9	12 + 134 + 2345
47	15	54321	9	12 + 134 + 135 + 234
48	16	54322	9	12 + 134 + 145 + 135 + 234 + 235

No.	V	$w_1 \sim w_6$	T	Representative Function
				$n = 6$
1	6	111111	4	1234 + 1235 + 1245 + 1345 + 1236 + 1246 + 1346 + 1256 + 1356 + 1456 + 2345 + 2346 + 2356 + 2456 + 3456
2	6	111111	5	12345 + 12346 + 12356 + 12456 + 13456 + 23456
3	6	111111	6	123456
4	7	211111	4	123 + 124 + 134 + 125 + 135 + 145 + 126 + 136 + 146 + 156 + 2345 + 2346 + 2356 + 2456 + 3456
5	7	211111	5	1234 + 1235 + 1245 + 1345 + 1236 + 1246 + 1346 + 1256 + 1356 + 1456 + 23456
6	7	211111	6	12345 + 12346 + 12356 + 12456 + 13456
7	8	221111	5	123 + 124 + 125 + 1345 + 126 + 1346 + 1356 + 1456 + 2345 + 2346 + 2356 + 2456
8	8	221111	6	1234 + 1235 + 1245 + 1236 + 1246 + 1256 + 13456 + 23456
9	8	221111	7	12345 + 12346 + 12356 + 12456
10	8	311111	5	123 + 124 + 134 + 125 + 135 + 145 + 126 + 136 + 146 + 156 + 23456
11	8	311111	6	1234 + 1235 + 1245 + 1345 + 1236 + 1246 + 1346 + 1256 + 1356 + 1456
12	9	222111	5	123 + 124 + 134 + 125 + 135 + 126 + 136 + 1456 + 234 + 235 + 236 + 2456 + 3456
47	11	521111	7	12 + 134 + 135 + 145 + 136 + 146 + 156
48	12	322221	7	123 + 124 + 134 + 125 + 145 + 135 + 2345 + 2346 + 2356 + 2456 + 3456
49	12	322221	8	1234 + 1235 + 1245 + 1345 + 1236 + 1246 + 1346 + 1256 + 1356 + 1456 + 2345
50	12	322221	9	1234 + 1235 + 1245 + 1345 + 23456
51	12	332211	7	123 + 124 + 134 + 125 + 126 + 1356 + 1456 + 234 + 2356 + 2456
52	12	332211	8	123 + 124 + 1345 + 1346 + 1256 + 2345 + 2346
53	12	332211	9	1234 + 1235 + 1245 + 1236 + 1246 + 13456 + 23456
54	12	332211	10	1234 + 12356 + 12456
55	12	333111	7	123 + 124 + 134 + 125 + 135 + 126 + 136 + 234 + 235 + 236
56	12	333111	9	123 + 12456 + 13456 + 23456

No.	V	$w_1 \sim w_5$	T	Representative Function
2	5	11111	4	1234 + 1235 + 1245 + 1345 + 2345
3	5	11111	5	12345
4	6	21111	4	123 + 124 + 134 + 125 + 135 + 145 + 2345
5	6	21111	5	1234 + 1235 + 1245 + 1345
6	7	22111	4	12 + 134 + 234 + 135 + 235 + 145 + 245
7	7	22111	5	123 + 124 + 125 + 1345 + 2345
8	7	22111	6	1234 + 1235 + 1245
9	7	31111	4	12 + 13 + 14 + 15 + 2345
10	7	31111	5	123 + 124 + 134 + 125 + 135 + 145
11	8	22211	5	123 + 124 + 134 + 234 + 125 + 135 + 235
12	8	22211	6	123 + 1245 + 1345 + 2345
13	8	22211	7	1234 + 1235
14	8	32111	5	12 + 134 + 135 + 145 + 2345
15	8	32111	6	123 + 124 + 125 + 1345
16	8	41111	5	12 + 13 + 14 + 15
17	9	32211	5	12 + 13 + 234 + 235 + 145
18	9	32211	6	123 + 124 + 134 + 125 + 135 + 2345
44	14	43322	8	123 + 124 + 125 + 134 + 135 + 145 + 234 + 235
45	14	53321	8	12 + 13 + 145 + 234
46	14	54221	9	12 + 134 + 2345
47	15	54321	9	12 + 134 + 135 + 234
48	16	54322	9	12 + 134 + 145 + 135 + 234 + 235

No.	V	$w_1 \sim w_6$	T	Representative Function
				$n = 6$
1	6	111111	4	1234 + 1235 + 1245 + 1345 + 1236 + 1246 + 1346 + 1256 + 1356 + 1456 + 2345 + 2346 + 2356 + 2456 + 3456
2	6	111111	5	12345 + 12346 + 12356 + 12456 + 13456 + 23456
3	6	111111	6	123456
4	7	211111	4	123 + 124 + 134 + 125 + 135 + 145 + 126 + 136 + 146 + 156 +2345+2346 + 2356 + 2456 + 3456
5	7	211111	5	1234 + 1235 + 1245 + 1345 + 1236 + 1246 + 1346 + 1256 + 1356 + 1456 + 2345
6	7	211111	6	12345 + 12346 + 12356 + 12456 + 13456
7	8	221111	5	123 + 124 + 125 + 1345 + 126 + 1346 + 1356 + 1456 + 2345 + 2346 + 2356 + 2456
8	8	221111	6	1234 + 1235 + 1245 + 1236 + 1246 + 1256 + 13456 + 23456
9	8	221111	7	12345 + 12346 + 12356 + 12456
10	8	311111	5	123 + 124 + 134 + 125 + 135 + 145 + 126 + 136 + 146 + 156 +23456
11	8	311111	6	1234 + 1235 + 1245 + 1345 + 1236 + 1246 + 1346 + 1256 + 1356 + 1456
12	9	222111	5	123 + 124 + 134 + 125 + 135 + 126 + 136 + 1456 + 234 + 235 + 236 + 2456 + 3456
13	9	222111	6	123 + 1245 + 1345 + 1246 + 1346 + 1256 + 1356 + 2345 + 2346 + 2356
14	9	222111	7	1234 + 1235 + 1236 + 12456 + 13456 + 23456
15	9	222111	8	12345 + 12346 + 12356
16	9	321111	5	12 + 134 + 135 + 145 + 136 + 146 + 156 + 2345 + 2346 + 2356 + 2456
17	9	321111	6	123 + 124 + 125 + 1345 + 126 + 1346 + 1356 + 1456 + 23456
18	9	321111	7	1234 + 1235 + 1245 + 1236 + 1246 + 1256 + 13456
19	9	411111	5	12 + 13 + 14 + 15 + 16 + 23456
20	9	411111	6	123 + 124 + 134 + 125 + 135 + 145 + 126 + 136 + 146 + 156
21	10	222211	6	123 + 124 + 134 + 1256 + 1356 + 1456 + 234 + 2356 + 2456 + 3456
22	10	222211	7	1234 + 1235 + 1245 + 1345 + 1236 + 1246 + 1346 + 2345 + 2346
23	10	222211	8	1234 + 12356 + 12456 + 13456 + 23456
24	10	222211	9	12345 + 12346
25	10	322111	6	123 + 124 + 134 + 125 + 135 + 126 + 136 + 1456 + 2345 + 2346 + 2356
26	10	322111	7	123 + 1245 + 1345 + 1246 + 1346 + 1256 + 1356 + 23456
27	10	322111	8	1234 + 1235 + 1236 + 12456 + 13456
28	10	331111	6	12 + 1345 + 1346 + 1356 + 1456 + 2345 + 2346 + 2356 + 2456
29	10	331111	7	123 + 124 + 125 + 126 + 13456 + 23456
30	10	331111	8	1234 + 1235 + 1245 + 1236 + 1246 + 1256
31	10	421111	6	12 + 134 + 135 + 145 + 136 + 146 + 156 + 23456
32	10	421111	7	123 + 124 + 125 + 1345 + 126 + 1346 + 1356 + 1456
33	10	511111	6	12 + 13 + 14 + 15 + 16
34	11	322211	6	123 + 124 + 134 + 125 + 135 + 145 + 126 + 136 + 146 + 234 + 2356 + 2456 + 3456
35	11	322211	7	123 + 124 + 134 + 1256 + 1356 + 1456 + 2345 + 2346
36	11	322211	8	1234 + 1235 + 1245 + 1345 + 1236 + 1246 + 1346 + 23456
37	11	322211	9	1234 + 12356 + 12456 + 13456
38	11	332111	6	12 + 134 + 135 + 136 + 1456 + 234 + 235 + 236 + 2456
39	11	332111	7	123 + 124 + 125 + 1345 + 126 + 1346 + 1356 + 2345 + 2346 + 2356
40	11	332111	8	123 + 1245 + 1246 + 1256 + 13456 + 23456
41	11	332111	9	1234 + 1235 + 1236 + 12456
42	11	422111	6	12 + 13 + 145 + 146 + 156 + 2345 + 2346 + 2356
43	11	422111	7	123 + 124 + 134 + 125 + 135 + 126 + 136 + 1456 + 23456
44	11	422111	8	123 + 1245 + 1345 + 1246 + 1346 + 1256 + 1356
45	11	431111	7	12 + 1345 + 1346 + 1356 + 1456 + 23456
46	11	431111	8	123 + 124 + 125 + 126 + 13456
47	11	521111	7	12 + 134 + 135 + 145 + 136 + 146 + 156
48	12	322221	7	123 + 124 + 134 + 125 + 145 + 135 + 2345 + 2346 + 2356 + 2456 + 3456
49	12	322221	8	1234 + 1235 + 1245 + 1345 + 1236 + 1246 + 1346 + 1256 + 1356 + 1456 + 2345
50	12	322221	9	1234 + 1235 + 1245 + 1345 + 23456
51	12	332211	7	123 + 124 + 134 + 125 + 126 + 1356 + 1456 + 234 + 2356 + 2456
52	12	332211	8	123 + 124 + 1345 + 1346 + 1256 + 2345 + 2346
53	12	332211	9	1234 + 1235 + 1245 + 1236 + 1246 + 13456 + 23456
54	12	332211	10	1234 + 12356 + 12456
55	12	333111	7	123 + 124 + 134 + 125 + 135 + 126 + 136 + 234 + 235 + 236
56	12	333111	9	123 + 12456 + 13456 + 23456
57	12	333111	10	1234 + 1235 + 1236
58	12	422211	7	123 + 124 + 134 + 125 + 135 + 145 + 126 + 136 + 146 + 2345 + 2346
59	12	422211	8	123 + 124 + 134 + 1256 + 1356 + 1456 + 23456
60	12	422211	9	1234 + 1235 + 1245 + 1345 + 1236 + 1246 + 1346
61	12	432111	7	12 + 134 + 135 + 136 + 1456 + 2345 + 2346 + 2356
62	12	432111	8	123 + 124 + 125 + 1345 + 126 + 1346 + 1356 + 23456
63	12	432111	9	123 + 1245 + 1246 + 1256 + 13456
64	12	441111	8	12 + 13456 + 23456
65	12	441111	9	123 + 124 + 125 + 126
66	12	522111	7	12 + 13 + 145 + 146 + 156 + 23456
67	12	522111	8	123 + 124 + 134 + 125 + 135 + 126 + 136 + 1456
68	12	531111	8	12 + 1345 + 1346 + 1356 + 1456
69	13	332221	7	123 + 124 + 134 + 125 + 135 + 145 + 126 + 234 + 235 + 245 + 3456
70	13	332221	8	123 + 124 + 125 + 1345 + 1346 + 1356 + 1456 + 2345 + 2346 + 2356 + 2456
71	13	332221	9	1234 + 1235 + 1245 + 1345 + 1236 + 1246 + 1256 + 2345
72	13	332221	10	1234 + 1235 + 1245 + 13456 + 23456
73	13	333211	8	123 + 124 + 134 + 1256 + 1356 + 234 + 2356
74	13	333211	9	123 + 1245 + 1345 + 1246 + 1346 + 2345 + 2346
75	13	333211	11	1234 + 12356
76	13	432211	7	12 + 134 + 135 + 145 + 136 + 146 + 234 + 2356 + 2456
77	13	432211	8	123 + 124 + 134 + 125 + 126 + 1356 + 1456 + 2345 + 2346
78	13	432211	9	123 + 124 + 1345 + 1346 + 1256 + 23456
79	13	432211	10	1234 + 1235 + 1245 + 1236 + 1246 + 13456
80	13	433111	7	12 + 13 + 1456 + 234 + 235 + 236
81	13	433111	8	123 + 124 + 134 + 125 + 135 + 126 + 136 + 2345 + 2346 + 2356
82	13	433111	10	123 + 12456 + 13456
83	13	442111	8	12 + 1345 + 1346 + 1356 + 2345 + 2346 + 2356
84	13	442111	10	123 + 1245 + 1246 + 1256
85	13	522211	7	12 + 13 + 14 + 156 + 2345 + 2346
86	13	522211	8	123 + 124 + 134 + 125 + 135 + 145 + 126 + 136 + 146 + 23456
87	13	522211	9	123 + 124 + 134 + 1256 + 1356 + 1456
88	13	532111	8	12 + 134 + 135 + 136 + 1456 + 23456
89	13	532111	9	123 + 124 + 125 + 1345 + 126 + 1346 + 1356
90	13	541111	9	12 + 13456
91	13	622111	8	12 + 13 + 145 + 146 + 156

TABLE 2
The Number of Majority Decision Functions

n	Number of Logical Functions of up to n Variables	Number of Types of Logical Functions of n Variables*	Number of Types of Majority Decision Functions of n Variables	Number of Majority Decision Functions of n Variables	Number of Types of Self-Dual Majority Decision Functions of n Variables
1	4	1	1	2	1
2	16	2	1	8	0
3	256	10	3	72	1
4	65, 536	208	9	1, 536	1
5	4, 294, 967, 296	615, 904	48	86, 080	4
6	18, 446, 774, 073, 709, 551, 616	—	504	14, 487, 040	14

* These values are obtained from the results in References [4] and [5].

TABLE 3
The Maximum Values of Optimum Parameters of Majority Decision Functions

n	w	$V = \sum_{i=1}^{n} w_i$	T	K
2	1	2	2	3
3	2	4	3	5
4	3	8	5	9
5	5	16	9	17
6	9	33	18	35

variables the solution space of (10) is a pointed cone. That is, there is a certain point x_0 such that

(11) $$Ax_0 \geqq b$$

and any solution x of (10) can be written as

(12) $$x = x_0 + x' \qquad Ax' \geqq 0.$$

This means the solution space of (10) is a cone with x_0 as a sole vertex. These structures for majority decision functions of six variables were examined and it was found that almost all the majority decision functions have solution space of a pointed cone but that 15 out of 504 representatives have spaces of non-cone structure. These functions are marked with * in Table 1.

Fourth, some maximum values of the optimum parameters are shown in Table 3, where V is the sum of coupling weights associated with input variables and K is the total number of turns of windings including the constant winding and the relation $K = 2T - 1$ holds. In Table 3, 26 functions have the maximum value 9 for a weight w and only one function attains the maximum value 33 of V; there are 7 functions with maximum K of 35.

7. Acknowledgment. The authors wish to express their thanks to Mr. R. O. Winder, RCA Laboratories, Princeton, New Jersey, for his courtesy in comparing his data with ours, and to Dr. S. Takasu, Electrical Communication Laboratory, Tokyo, for his stimulating discussions.

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Received February 19, 1