THE GROWTH OF DIGITAL SUMS OF POWERS OF TWO

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In this note, we give an elementary proof that \( s(2^n) > \log_4 n \) for all \( n \), where \( s(n) \) denotes the sum of the digits of \( n \) written in base 10. In particular, \( \lim_{n \to \infty} s(2^n) = \infty \).

The reader will notice that the lower bound is very weak. The number of digits of \( 2^n \) is \( \lfloor n \log_{10} 2 \rfloor + 1 \), so it is natural to conjecture that

\[
\lim_{n \to \infty} \frac{s(2^n)}{n} = 4.5 \log_{10} 2.
\]

However, this conjecture remains open\(^2\).

In 1970, H. G. Senge and E. G. Strauss proved that the number of integers whose sum of digits is bounded with respect to the bases \( a \) and \( b \) is finite if and only if \( \log_b a \) is rational\(^3\). Of course the sum of the digits of \( a^n \) in base \( a \) is 1, so this result implies that

\[
\lim_{n \to \infty} s(a^n) = \infty
\]

for all positive integers \( a \) except powers of 10. This work was extended by C. L. Stewart, who gave an effectively computable lower bound for \( s(a^n) \) \(^3\). However, this lower bound is weaker than ours, and Stewart’s proof relies on deep results in transcendental number theory.

We begin with two simple lemmas.

**Lemma 1.** Every positive integer \( N \) can be expressed in the form

\[
N = \sum_{i=1}^{m} d[i] \cdot 10^{e[i]}
\]

where \( d[i] \) and \( e[i] \) are integers so that \( 1 \leq d[i] \leq 9 \) and

\[
0 \leq e[1] < e[2] < \cdots < e[m]
\]

Furthermore,

\[
s(N) = \sum_{i=1}^{m} d[i] \geq m
\]

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Proof. The proof is by strong induction on $N$. The case $N < 10$ is trivial. Suppose that $N \geq 10$. By the division algorithm, there exist integers $n \geq 1$ and $0 \leq r \leq 9$ so that $N = 10n + r$. By the induction hypothesis, we can express $n$ in the form

$$n = \sum_{i=1}^{m} d[i] \cdot 10^{e[i]}$$

If $r = 0$, then

$$N = \sum_{i=1}^{m} d[i] \cdot 10^{e[i]+1}$$

and if $r > 0$ then

$$N = r \cdot 10^0 + \sum_{i=1}^{m} d[i] \cdot 10^{e[i]+1}$$

In either case, $N$ has an expression of the required form. \hfill \Box

Lemma 2. Let $2^n = A + B \cdot 10^k$ where $A, B, k, n$ are positive integers and $A < 10^k$. Then $A \geq 2^k$.

Proof. Since $2^n > 10^k > 2^k$, it follows that $n > k$, so $2^k$ divides $2^n$. But $2^k$ also divides $10^k$, therefore $2^k$ divides $A$. But $A > 0$, so $A \geq 2^k$. \hfill \Box

We use these lemmas to establish a lower bound on $s(2^n)$. Write

$$2^n = \sum_{i=1}^{m} d[i] \cdot 10^{e[i]}$$

so the conditions of Lemma 1 hold, and let $k$ be an integer between 2 and $m$. Then $2^n = A + B \cdot 10^{e[k]}$ where

$$A = \sum_{i=1}^{k-1} d[i] \cdot 10^{e[i]}$$

and

$$B = \sum_{i=k}^{m} d[i] \cdot 10^{e[i]-e[k]}$$

Since $A < 10^{e[k]}$, Lemma 2 implies that $A \geq 2^{e[k]}$. Therefore,

$$2^{e[k]} \leq A < 10^{e[k-1]+1}$$

which implies that

$$e[k] \leq \lfloor (\log_2 10)(e[k-1] + 1) \rfloor$$

We prove that $e[k] < 4^{k-1}$ for all $k$. It is clear that $e[1] = 0$, else $2^n$ would be divisible by 10. From the inequality above, we have $e[1] \leq 3$, $e[2] \leq 13$,
$e[3] \leq 46, e[4] \leq 156, e[5] \leq 521, \text{and } e[6] \leq 1734$. If $k \geq 7$ then $e[k-1] \geq 5$, so

$$
e[k] < (\log_2 10)e[k-1] + (\log_2 10)
\leq \frac{10}{3}e[k-1] + \frac{10}{3}
\leq \frac{10}{3}e[k-1] + \frac{2}{3}e[k-1]
= 4e[k-1]$$

Therefore, $e[k] < 4^{k-1}$ for all $k$, by induction.

We are now able to prove the main result. Note that

$$2^n < 10^{e[m]+1} \leq 10^{4^{m-1}}$$

since $10^{e[m]}$ is the leading power of 10 in the decimal expansion of $2^n$.

Taking logarithms gives

$$4^{m-1} > n \log_{10} 2
4^{m-1} > n/4
4^m > n
m > \log_4 n
s(2^n) > \log_4 n$$

hence

$$\lim_{n \to \infty} s(2^n) = \infty$$

References