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1. Introduction

The cell growth problem for square celled animals is illustrated in Figure 1 which shows all $a_n = 5$ such animals with just 4 cells.

The problem is to find some sort of exact formula for $a_n$. One of us has offered $100 for the first solution and this offer is still outstanding, as noted in [1,2].

Figure 1. The 5 animals with 4 square cells.

The variation of this problem which we are able to handle is to count "tapeworms." A measure of an animal's "complexity" is its skeleton, that is, the graph obtained by replacing each cell by a point, with cells containing a common edge corresponding to adjacent points in the skeleton.

A filament is an animal whose skeleton is a path. Even these are hard to count because of the restriction that they can be drawn in the plane.

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Figure 2. The 4 tapeworms with 4 cells, up to orientation and direction

A **tapeworm** is just like a filament except that it doesn't have to be planar. A tapeworm may be grown one cell at a time, starting with the head. The essential restriction is that no three cells have a common edge. The 4-celled tapeworms are illustrated in Figure 2. Two cells are adjacent just if their labels are consecutive.

A tapeworm may be **oriented** by assigning a positive direction to the perimeter of one cell and compatible directions (opposite at any common edge) to adjacent cells.

A tapeworm is **directed** by choosing one of the end cells to be the head. If there are at least two cells, there is one other end cell which we call the tail. With each oriented directed tapeworm we associate a sequence of numbers from \([-1, 0, 1]\). The \(n\)th number of the sequence is 0 if the tapeworm makes no turn at the \(n\)th cell from the head, 1 if the turn is in the positive direction (traversing the worm from head to tail), and -1 if the turn is negative.
It is easy to visualize the assignment of $+1$, $0$, or $-1$ to a cell by putting it and its neighbors in the plane in such a way that positive orientation is counterclockwise. Then, traversing the tapeworm from head to tail, assign $+1$ for a right hand turn, $-1$ for a left hand turn, and $0$ for no turn. To illustrate, consider the worms of Figure 2 to be directed by choosing cell $1$ as the head, and oriented so that the positive direction for each cell is counterclockwise. Then the sequences for these tapeworms are $(0,0)$, $(0,-1)$, $(-1,1)$, and $(-1,-1)$ respectively. Notice that the end cells are not represented by a number in the associated sequence. Let $q_n$ be the number of distinct oriented directed tapeworms with exactly $n$ cells. Since the sequences of length $n$ are in 1-1 correspondence with the oriented directed worms with $n+2$ cells, we have an obvious formula for $q_{n+2}$.

**Theorem 1.** The number of oriented directed tapeworms is given by

$$q_{n+2} = 3^n \quad \text{for } n \geq 0.$$  

In [3] Tilley, Stanton, and Cowan counted tapeworms with the property that when oriented and directed, the associated sequence contains no two consecutive $1$'s or $-1$'s. This has the merit of forbidding the most common cause for a tapeworm not to be planar, as illustrated by the last worm of Figure 2. If $l_n$ is the number of oriented directed tapeworms with $n$ cells satisfying
this restriction, then it is shown in [3] that

\[ \lambda_n = \frac{1}{2} (1 + \sqrt{2})^{n+1} + \frac{1}{2} (1 - \sqrt{2})^{n+1}. \]

If \( f_n \) is the corresponding number for tapeworms without orientation or direction, then

\[ f_{2n+1} = \frac{1}{4} (\lambda_{2n-1} + \lambda_n + \lambda_{n-1} + 1) \]
\[ f_{2n+2} = \frac{1}{4} (\lambda_{2n} + \lambda_n + \lambda_{n-1} + 1) \]

for \( n \geq 1 \).

Returning to the larger question of unrestricted tapeworms, let \( r_n \) be the number of distinct directed tapeworms with \( n \) cells, and \( t_n \) the corresponding number of free tapeworms (neither oriented nor directed).

**Theorem 2.** The number of directed tapeworms is given by

\[ r_{n+2} = \frac{q_{n+2} + 1}{2} = \frac{3^n + 1}{2}, \quad n \geq 0. \]

**Proof.** Reversing the orientation of the worm \( a_1, \ldots, a_n \) gives the worm \((-a_1, -a_2, \ldots, -a_n)\). Thus the straight worm \((0, 0, \ldots, 0)\) is the only one which is isomorphic to its re-orientation.

**Theorem 3.** For \( n \geq 0 \), the number of tapeworms is given by

\[ t_{2n+2} = \left( \frac{3^n + 1}{2} \right)^2 \]
\[ t_{2n+3} = \frac{3^{2n+1} + 1}{4} + 3^n \]

**Proof.** Reversing the direction of the oriented directed tapeworm \((a_1, a_2, \ldots, a_{2n})\)
gives \((a_{2n}, \ldots, a_2, a_1)\). The reversed tapeworm may be isomorphic to the original with orientation either preserved or reversed. Thus in the even case the reversal-symmetric worms are either of type:

\[(a_1, a_2, \ldots, a_n, a_n, \ldots, a_2, a_1)\]

or \[(a_1, a_2, \ldots, a_n, -a_n, \ldots, -a_2, -a_1)\].

The straight worm is the only one of both types. Therefore \(\frac{3^{n+1}}{2} \cdot 2 - 1 = 3^n\) directed tapeworms are reversal symmetric, and so

\[t_{2n+2} = \frac{r_{2n+2}}{2} + \frac{3^n}{4} + \frac{3^n}{2} = \left(\frac{3^{n+1}}{2}\right)^2.\]

In the odd case, the reversal symmetric tapeworms are of type

\[(a_1, \ldots, a_n, a_{n+1}, a_n, \ldots, a_1)\]

or \[(a_1, \ldots, a_n, 0, -a_n, \ldots, -a_1)\],

the straight worm being the only one of both types. There are \(\frac{3^{n+1}+1}{2}\) tapeworms of the first type and \(\frac{3^{n+1}}{2}\) of the second, so in all

\[\frac{3^{n+1}+3^n}{2} = 2 \cdot 3^n\] have end reversal symmetry. Thus

\[t_{2n+3} = \frac{r_{2n+3}}{2} + \frac{2 \cdot 3^n}{2} = \frac{3^{2n+1}+1}{4} + 3^n.\]

Note that asymptotically, \(t_{n+1} \sim \frac{3^n}{12}\)

while \(f_{n+1} \sim \frac{(1+\sqrt{2})^n}{8}\).
References


The University of Michigan