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# Folding a Strip of Stamps*

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## ABSTRACT

This paper answers the problem of determining how many ways a strip of $N$ stamps can be folded along the perforations so that the stamps are piled one on top of the other without destroying the continuity of the strip. Using combinatorial arguments, the problem is first solved by an inductive formula for $N$ odd. Part of the computations involved are seen to yield an answer to the case when $N$ is even.

Three cases are considered: (1) both, (2) one of, (3) neither, the left end and top of the strip are labeled. If $P(N)$ is the number of folds for a strip of $N$ stamps of type (1), then $\frac{1}{2}P(N)$ and $\frac{1}{4}P(N) + \frac{1}{4}S(N)$ yields formulas for the number of folds for a strip of $N$ stamps of types (2) and (3), where $S(N)$ is the number of symmetric folds of the strip.

$P(N)$ also equals the number of ways of joining $N$ dots on a circle by chords of alternating blue and red color without having any chords of the same color intersect.

The formulas for $P(N)$ and $S(N)$ are too complicated to give here. However, the first few values are listed by computer as:

| $N$ | $P(N)$ | $S(N)$ | $N$ | $P(N)$ | $S(N)$ |
|---|---|---|---|---|---|
| 3 | 6 | 2 | 10 | 14060 | 48 |
| 4 | 16 | 4 | 11 | 46310 | 178 |
| 5 | 50 | 6 | 12 | 146376 | 132 |
| 6 | 144 | 8 | 13 | 485914 | 574 |
| 7 | 462 | 18 | 14 | 1557892 | 348 |
| 8 | 1392 | 20 | 15 | 5202690 | 1870 |
| 9 | 4536 | 56 | 16 | 16861984 | 1008 |


## 1. THE PROBLEM: FOLDS AND PATTERNS

1.1 PRELIMINARIES. Assume for the present that a strip of $N$ stamps has a definite back and front side and left and right edge. Number the stamps in order, 1. 2,..., $N$, beginning from the left front. The problem arises of determining how many ways the strip can be folded along the

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perforations so that the stamps are piled one of top of the other, without destroying the continuity of the strip. (Later we shall consider the cases when one or both of the restrictions on the strip are removed.)

Call such a pile of $N$ stamps an *N-fold*. Assume each $N$-fold to be oriented so that stamp 1 has its front side facing the top of the pile, its left edge on the left as we look down on the pile. Thus we may picture an $N$-fold by a continuous bent line, as in Figure 1, where the vertical

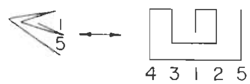


FIGURE 1

sections represent the stamps and the horizontal sections represent the perforations. We may describe an $N$-fold by listing the stamps as they appear, beginning from the top of the pile. For example, the 5-fold of Figure 1 may be described by 4, 3, 1, 2, 5. Figure 2 lists the sixteen 4-folds.

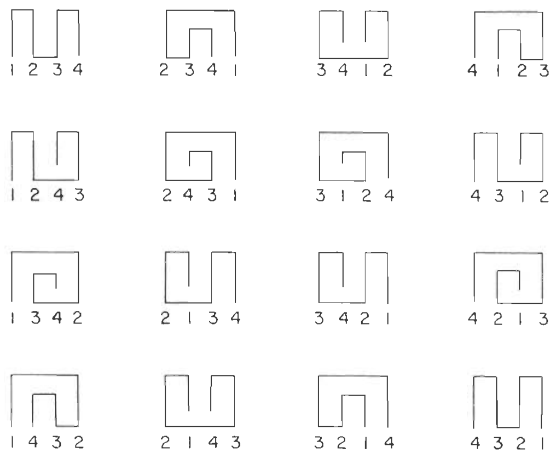


FIGURE 2

Two properties of this pictorial representation are evident:

(A) The line cannot cross itself. For this would mean that the upper and lower edges of the strip touch at the point of intersection; in other words, the strip would be in a tubular roll rather than a pile.

(B) The horizontal segments representing the perforations between the odd numbered stamps and their immediate successors lie above the vertical segments, whereas the segments representing the perforations between the even numbered stamps and their successors lie below the vertical segments. This property follows by induction on $N$.

Property $A$ must be violated if and only if between some stamp $k$ and its successor $k + 1$ there is either a stamp $r$ whose successor $r + 1$ lies outside the pair $k, k + 1$ in the ordering, or a stamp $r + 1$ whose predecessor $r$ lies outside the pair (see Fig. 3.) Property B implies that $k$

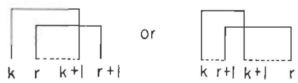


FIGURE 3

and $r$ are both even or both odd. There are eight such situations for a given $k$ and $r$:

$$\begin{array}{llll} k, & r, & k+1, & r+1 \\ r, & k+1, & r+1, & k \\ k+1, & r+1, & k, & r \\ r+1, & k, & r, & k+1 \end{array}$$

and four more where the roles of $k$ and $r$ are interchanged.

1.2. THEOREM. *An ordering of the numbers* 1, 2,..., $N$ *describes an N-fold if and only if*

(1) *No numbers appear in the order* $\ldots k \ldots r \ldots k+1 \ldots r+1$, *or any of the circular permutations of these numbers, where* $1 \leqslant k, r \leqslant N-1$; $k \neq r$; $k \equiv r \pmod 2$.

PROOF: The necessity of (1) is evident from the paragraph above. Sufficiency may be shown by induction on $N$. For every $N$-fold may be considered as a superfold of the stamps 1, 2,..., $N - 1$, with stamp $N$ so placed as to make a fold. By the induction hypothesis, we may assume that all the $(N - 1)$-folds are described by all the possible orderings of $N - 1$ numbers satisfying (1).

If the vertical line representing stamp $N - 1$ when extended would hit a horizontal line representing the perforations between some stamp $k$ and $k + 1$, where $k \equiv N - 1 \pmod 2$, then, to make a fold, the horizontal line of perforations between $N - 1$ and $N$ must lie completely within the other horizontal line, and cannot cross the vertical lines of stamps $k$ or $k + 1$.

If the vertical line of stamp $N - 1$ when extended misses the horizontal line of perforations between $k$ and $k + 1$, $k \equiv N - 1 \pmod 2$, then the vertical line of stamp $N$ may be placed so that stamps $k$, $k+1$ lie either entirely within or entirely without the lines $N$, $N-1$.

But these possibilities for placing stamp $N$ are precisely all the ways of satisfying (1). Q.E.D.

1.3. Definition. We say that four integers $k_1, k_2, k_3, k_4$ are in circular order, and we write $k_1 \to k_2 \to k_3 \to k_4$, if $k_1 < k_2 < k_3 < k_4$ or $k_2 < k_3 < k_4 < k_1$ or $k_3 < k_4 < k_1 < k_2$ or $k_4 < k_1 < k_2 < k_3$.

Throughout, let $J_N = \{1, 2, \ldots, N\}$.

1.4. Definition. A permutation $\pi$ of $J_N$ is called an $N$-fold if

(2) $\pi(k) \to \pi(r) \to \pi(k+1) \to \pi(r+1)$ does not occur when $k$ and $r$ are either both odd or both even.

An $N$-fold $\pi$ is said to be oriented if $\pi(1) = 1$.

Remark. To each $N$-fold there corresponds a permutation $\pi$ of $J_N$ defined by $\pi(k) = x_k$, where stamp $k$ is in tye $x_k$-th position from the top of the pile, $k \in J_N$. Clearly if the stamps appear in the order $\ldots k \ldots r \ldots k+1 \ldots r+1 \ldots$ or any of the circular permutations of these numbers, then $\pi(k) \to \pi(r) \to \pi(k+1) \to \pi(r+1)$, and conversely. Thus 1.4(2) is equivalent to 1.2(1), and so our definitions of an $N$-fold are consistent.

To each oriented $N$-fold $\pi$, there corresponds a unique $N$-cycle $(1, \pi(2), \ldots, \pi(N))$ which we shall also call $\pi$.

1.5. Definition. A shifting function $\sigma$ is a function from $J_N$ into $J_N$ such that

(i) $\sigma$ is one-to-one,

(ii) $\sigma$ preserves circular order; that is, if $k_1 \to k_2 \to k_3 \to k_4$ and all $k_i$ are in $D(\sigma)$, the domain of $\sigma$, then

$$\sigma(k_1) \to \sigma(k_2) \to \sigma(k_3) \to \sigma(k_4).$$

Remark. Let $\sigma$ be a shifting function. Then, since $\sigma$ is one-to-one and circular order preserving, $\sigma$ also preserves the length $K$ of all $K$-cycles within its domain. If $\sigma$ is on $J_N$, then $\sigma$ is also onto $J_N$. It is easy to show that $\sigma^{-1}$ is also a shifting function, because the assumption that (ii) does not hold for $\sigma^{-1}$ leads to a contradiction of that property for $\sigma$.

For each $t \in J_N$, define a function $\sigma_t$ on $J_N$ by

$$\sigma_t(k) = \begin{cases} k - t + 1 & \text{if } t \leqslant k \leqslant N, \\ \sigma_t(N) + k & \text{if } 1 \leqslant k \leqslant t - 1. \end{cases}$$

The $\sigma_t$ so defined are shifting functions, and are the only shifting functions on $J_N$. They have the additional properties: $\sigma_t^{-1} = \sigma_s \Leftrightarrow s + t \equiv 2 \pmod N$ and $\sigma_s \sigma_t = \sigma_r \Leftrightarrow r \equiv s + t - 1 \pmod N$.

1.6. THEOREM. *Let $P(N)$ be the number of $N$-folds; $P_0(N)$ the number of oriented $N$-folds; $P_2(N)$ the number of oriented $N$-folds $\pi$ such that $\pi(2) = 2$. Then* (1) $P(N) = NP_0(N)$,

(2) $P_0(N-1) = P_2(N)$,

(3) $P_0(N) \equiv 0 \pmod 2$ if $N > 2$.

$P(N)$
$P_0(N)$
$P_2(N)$

PROOF: $\pi$ is an $N$-fold,

$\Leftrightarrow \pi(k) \to \pi(r) \to \pi(k+1) \to \pi(r+1)$ does not occur,

$\Leftrightarrow \sigma_t\pi(k) \to \sigma_t\pi(r) \to \sigma_t\pi(k+1) \to \sigma_t\pi(r+1)$ does not occur,

$\Leftrightarrow \sigma_t\pi$ is an $N$-fold,

where $\sigma_t$ is any shifting function on $J_N$, $k, r \in J_N$, $k \equiv r \pmod 2$.

(1) By the properties of the $\sigma_t$, we may define an equivalence relation on $N$-folds by $\pi \equiv \pi'$ if $\pi' = \sigma_t\pi$ for some $t$. We have just showed that each equivalence class has exactly $N$ members, since $\sigma_s\pi = \sigma_t\pi$ only if $s = t$. Further, if if $\pi(1) = k$, then $\sigma_t\pi$ is oriented $\Leftrightarrow \sigma_t\pi(1) = \sigma_t(k) = 1 \Leftrightarrow t = k$. Thus exactly one member of each equivalence class is oriented. Therefore $P(N) = NP_0(N)$.

(2) If $\pi$ is an oriented $N$-fold such that $\pi(2) = 2$, then $\sigma_2\pi\sigma_2^{-1}$ restricted to $J_{N-1}$ is an oriented $(N-1)$-fold. Conversely, if $\pi'$ is an oriented $(N-1)$-fold, then $\pi$ defined by $\pi(1) = 1, \pi(k) = \pi'(k-1) + 1 = \sigma_2^{-1}\pi'\sigma_2(k)$ is an oriented $N$-fold such that $\pi(2) = 2$. Hence $P_0(N-1) = P_2(N)$.

(3) Let $\pi$ be an $N$-fold. Define $\pi'$ by $\pi'(k) = \pi(N+1-k)$, $k \in J_N$. Then

$\pi(k) \to \pi(r) \to \pi(k+1) \to \pi(r+1)$ does not occur,

$\Leftrightarrow \pi(k+1) \to \pi(r+1) \to \pi(k) \to \pi(r)$ does not occur,

$\Leftrightarrow \pi'(s) \to \pi'(t) \to \pi'(s+1) \to \pi'(t+1)$ does not occur,

where $s = N - k$, $t = N - r$, so that $k - r \equiv s - t \pmod 2$. Hence $\pi'$ is an $N$-fold. This correspondence $\pi \Leftrightarrow \pi'$ pairs $N$-folds uniquely. Hence $P(N) \equiv 0 \pmod 2$. If $N > 2$, pairs $\pi, \pi'$ cannot be in the same equivalence class defined in (1); thus $P_0(N) \equiv 0 \pmod 2$.

Henceforth let $N$ be an odd integer greater than one unless we specify otherwise. Using Theorem 1.6, if we can find $P_0(N)$ and $P_2(N)$, then we can find $P(K)$ for every integer $K$.

The repuirements for an $N$-fold suggest breaking up the $N$-cycle into a product of two terms, each term consisting of a product of 2-cycles $(\pi(k), \pi(k+1))$, one term having $k$ always odd, the other having $k$ even.

1.7. Definition. Let $N$ be an odd integer. A permutation $\alpha$ of $J_N$ is called an $N$-pattern if, for all $k \in J_N$,

(i) $\alpha(\alpha(k)) = k$,

(ii) there is a unique $k_0 \in J_N$ such that $\alpha(k_0) = k_0$,

(iii) if $k < n < \alpha(k)$, then $k < \alpha(n) < \alpha(k)$.

An $N$-pattern $\alpha$ is said to be oriented if $\alpha(1) = 1$.

Let $\alpha$ and $\alpha'$ be two $N$-patterns. Let $\alpha \times \alpha'$ denote the product of the two permutations. Call $\alpha \times \alpha'$ oriented if $\alpha$ is oriented.

It is easy to show that (iii) is equivalent to

(iii)′ $k \to n \to \alpha(k) \to \alpha(n)$ does not occur, or to

(iii)″ $k \to n \to \alpha(n) \to \alpha(k)$ must occur, where $k, n \in J_N$.

1.8. Theorem. *To each oriented $N$-fold $\pi$ there corresponds a unique oriented $N$-cycle $\alpha \times \alpha'$, where $\alpha, \alpha'$ are $N$-patterns.*

Proof: Let $\pi$ be an oriented $N$-fold. Define $\alpha$ by $\alpha(1) = 1$; $\alpha(\pi(2k)) = \pi(2k+1)$; $\alpha(\pi(2k+1)) = \pi(2k)$. By Remark 1.4, $\pi(2k) \to \pi(2r) \to \pi(2k+1) \to \pi(2r+1)$ does not occur; thus $n \to m \to \alpha(n) \to \alpha(m)$ does not occur whenever $n, m \in J_N$. Thus $\alpha$ is an oriented $N$-pattern.

Define $\alpha'$ by $\alpha'(\pi(2k-1)) = \pi(2k)$; $\alpha'(\pi(2k)) = \pi(2k-1)$; $\alpha'(\pi(N)) = \pi(N)$. $\alpha'$ is also an $N$-pattern.

$$\begin{aligned}\alpha \times \alpha' &= \alpha(1)(\alpha'(1), \alpha'(2))(\alpha(2), \alpha(3)) \cdots (\alpha(N-1), \alpha(N))(\alpha'(N)) \\ &= (1)(\pi(2), \pi(1))(\pi(3), \pi(2)) \cdots (\pi(N-1), \pi(N))(\pi(N)) \\ &= (1, \pi(2), \pi(3), \ldots, \pi(N)) = \pi.\end{aligned}$$

The correspondence is unique, since an $N$-cycle is uniquely written as a product of 2-cycles, and these can be divided into two sets of disjoint 2-cycles in only one way.

Remark. We shall refer to an (oriented) $N$-cycle $\alpha \times \alpha'$ as an (oriented) $N$-fold. If we pick $N$ points on a circle and number them in order around the circle, the conditions on an $N$-pattern $\alpha$ are precisely those guaranteeing that the lines joining the pairs $k, \alpha(k)$ will not intersect. If $\alpha, \alpha'$ are $N$-patterns, then $\alpha \times \alpha'$ will be an $N$-fold if and only if all the $N$ points are joined by a continuous line (Fig. 4). An inductive argument [see 1, p. 354) shows that there are exactly

$$(N-1)!/[\tfrac{1}{2}(N-1)]![\tfrac{1}{2}(N+1)]!$$

oriented N-patterns $\alpha$, where $N$ is odd. The rest of the paper becomes clearer if this picture is kept in mind.

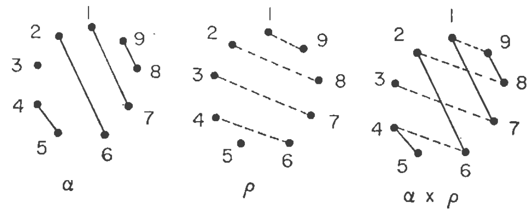


FIGURE 4

1.9. DEFINITION. An $N$-pattern $\alpha$ is said to have a short line at $k$ if $\alpha(k) = k + 1$, where $k \in J_{N-1}$, or if $\alpha(N) = 1$. We shall call a short line at $k$, $1 < k < N$, a mid line.

$\alpha$ is said to have a gap at $k$ if $\alpha$ does not have a short line at $k$.

1.10. LEMMA. *Every N-pattern has at least one short line, and at most* $(N-1)/2$ *short lines.*

PROOF: The second half of the assertion is obvious. For the first half, let $\alpha$ be an oriented $N$-pattern. Let $m = \min\{\alpha(k) - k \mid 1 \leqslant k < \alpha(k) \leqslant N\}$. If $m = 1$, we are done. If $m > 1$, then $k < k + 1 < \alpha(k)$ for each $k$. Hence $k + 1 < \alpha(k + 1) < \alpha(k)$ and $\alpha(k + 1) - (k + 1) \leqslant m - 2$ for some $k$, a contradiction. Hence $m = 1$ in all cases.

If $\alpha$ is any $N$-pattern, then $\alpha = \sigma_t\alpha'$, $\alpha'$ some oriented $N$-pattern. If $\alpha'$ has a short line at $k$, then $\alpha$ has a short line at $\sigma_t(k)$. This completes the proof.

In an oriented $N$-fold, if the points $k$ and $k + 1$ are joined by a short line, this means that the $k$-th and $k + 1$-st stamps in the pile are also adjacent stamps in the strip. If $1 < k < N$, then we could cut the perforations which join this adjacent pair of stamps to the rest of the strip, remove the pair, and rejoin the two remaining cut edges to form an oriented $(N - 2)$-fold. If $k = 1$ of if the points $N$ and 1 are joined by a short line, then we could simply remove stamps 1 and $\alpha'(1) = \pi(2)$ from the strip, leaving an $(N - 2)$-fold, not necessarily oriented. This removal process we now carry out in terms of the functions $\alpha$ and $\alpha'$. See Figure 5 for an illustration of these cases.

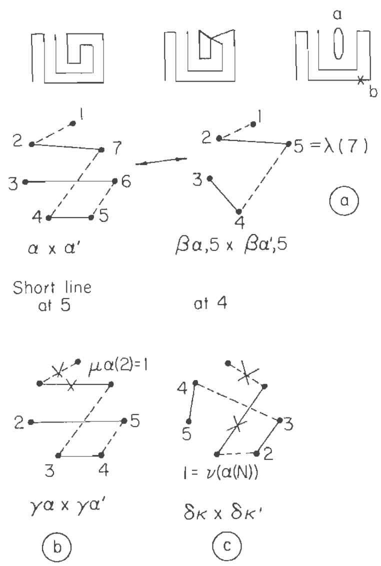


FIGURE 5

1.11 DEFINITION. For each integer $k$, define a function $\lambda_k$ by: $\lambda_k(n) = n$ if $n < k$, $\lambda_k(n) = n - 2$ if $n > k + 1$.

Let $\alpha$ be an $N$-pattern. Let $2 \leqslant k \leqslant N - 1$. If $\alpha$ has a short line at $k$, or if $\alpha$ is not oriented, let $\alpha_k = \alpha$. If $\alpha$ is oriented and has a gap at $k$, define $\alpha_k$ by:

$$\alpha_k(k) = k + 1, \qquad \alpha_k(k + 1) = k$$
$$\alpha_k(\alpha(k)) = \alpha(k + 1), \qquad \alpha_k(\alpha(k + 1)) = \alpha(k).$$
$$\alpha_k(n) = \alpha(n) \text{ for other } n \in J_N .$$

Finally, define a function $\beta_{\alpha,k}$ by

$$\beta_{\alpha,k}(n) = \lambda_k \alpha_k \lambda_k^{-1}(n), \qquad n \in J_{N-2} .$$

1.12. LEMMA. (1) *$\alpha_k$ is an $N$-pattern which is oriented if $\alpha$ is.*

(2) *$\lambda_k$ and $\lambda_k^{-1}$ are shifting functions.*

(3) *$\beta_{\alpha,k}$ is an $(N - 2)$-pattern which is oriented if $\alpha$ is.*

The proof is routine by checking the definitions.

1.13. LEMMA. *Let $\alpha$ be an $N$-pattern with $m$ short lines, of which one is at $k$, $2 \leqslant k < N$. Then $\beta_{\alpha,k}$ has $m$ short lines if $\alpha(k - 1) \equiv k + 2 \pmod N$, including one at $k - 1$. Otherwise $\beta_{\alpha,k}$ has $m - 1$ short lines, with none at $k - 1$.*

PROOF: $\beta_{\alpha,k}(k-1) = \lambda_k \alpha_k \lambda_k^{-1}(k-1) = \lambda_k \alpha \lambda_k^{-1}(k-1) = \lambda_k \alpha(k-1) = \lambda_k(k+2) = k$ if an only if $\alpha(k-1) = k+2$, where $2 \leqslant k < N-1$. Similarly, $\beta_{\alpha,k}(N-2) = 1$ if and only if $\alpha(N-2) = 1 \equiv (N-1)+2 \pmod{N}$.

Since shifting functions preserve all cycles in their domains, $\beta_{\alpha,k}$ will have a corresponding short line to every short line of $\alpha$, except for the short line at $k$, which will disappear. The lemma follows at once.

1.14. DEFINITION. Let $\alpha$ be an oriented $N$-pattern, $N \geqslant 5$. Define as follows:

(a) $\mu(n) = N + n - \alpha(2) - 1$ if $3 \leqslant n < \alpha(2)$,
$\mu(n) = n - \alpha(2) + 1$ if $\alpha(2) \leqslant n \leqslant N$.

(b) $\nu(n) = N + n - \alpha(N) - 1$ if $2 \leqslant n < \alpha(N)$,
$\nu(n) = n - \alpha(N) + 1$ if $\alpha(N) \leqslant n \leqslant N-1$.

(c) $\gamma_\alpha(1) = 1$, $\gamma_\alpha(n) = \mu\alpha\mu^{-1}(n)$, $n = 2, 3, \ldots, N-2$.

(d) $\delta_\alpha(1) = 1$, $\delta_\alpha(n) = \nu\alpha\nu^{-1}(n)$, $n = 2, 3, \ldots, N-2$.

(e) $\gamma_{\alpha'} = \mu\alpha'\mu^{-1}$, $\delta_{\alpha'} = \nu\alpha'\nu^{-1}$,

where $\alpha$ is fixed and oriented, and $\alpha'$ is any $N$-pattern.

1.15. LEMMA. *$\gamma_\alpha$ and $\delta_\alpha$ are $(N-2)$-patterns. $\gamma_{\alpha'}$ is an $(N-2)$-pattern if $\alpha'$ has a short line at* 1; *$\delta_{\alpha'}$ is an $(N-2)$-pattern if $\alpha'$ has a short line at $N$.*

The proof follows that of Lemma 1.12.

1.16. THEOREM. *Let $\alpha \times \alpha'$ be an oriented $N$-fold, where $\alpha$ has a gap at $k$, and $\alpha'$ has a short line at $k$, $2 \leqslant k \leqslant N-1$. Then $\beta_{\alpha,k} \times \beta_{\alpha',k}$ is an oriented $(N-2)$-fold. Conversely, if $\beta_{\alpha,k} \times \beta$ is an oriented $(N-2)$-fold, then there is a unique $\alpha'$ having a short line at $k$ such that $\beta = \beta_{\alpha',k}$ and $\alpha \times \alpha'$ is an $N$-fold.*

PROOF: For the first half of the theorem, the conditions guarantee that $\beta_{\alpha,k}$ is an oriented $(N-2)$-pattern and $\beta_{\alpha',k}$ is an $(N-2)$-pattern. It remains to show that $\beta_{\alpha,k} \times \beta_{\alpha',k}$ is an $(N-2)$-cycle.

Let $\alpha \times \alpha' = (k_1, k_2, \ldots, k_N) = (k_1, k_2)(k_2, k_3) \cdots (k_{N-1}, k_N)$. Somewhere among the 2-cycles appear $(k, \alpha'(k)) = (k, k+1)$, $(k, \alpha(k))$, and $(k+1, \alpha(k+1))$. Thus

$$\alpha \times \alpha' = (k_1, \ldots, k_r)(\alpha(k), k, k+1, \alpha(k+1))(k_{r+3}, \ldots, k_N) \quad \text{or}$$
$$\cdots(\alpha(k+1), k+1, k, \alpha(k))\cdots,$$

where $k_r = \alpha(k)$ and $k_{r+3} = \alpha(k+1)$ in the first case, and $k_r = \alpha(k+1)$ and $k_{r+3} = \alpha(k)$ in the second case.

$$\beta_{\alpha,k} \times \beta_{\alpha',k} = (\lambda_k(k_1),..., \lambda_k(k_r))(\lambda_k\alpha(k), \lambda_k\alpha(k+1))(\lambda_k(k_{r+3}),..., k_N) \quad \text{or}$$
$$\cdots (\lambda_k\alpha(k+1), \lambda_k\alpha(k)) \cdots$$
$$= (\lambda_k(k_1),..., \lambda_k(k_N)),$$

an $(N-2)$-cycle. This is true since $\lambda_k$ is a shifting function, and so preserves the length of the cycles within its domain.

Conversely, let $\beta_{\alpha,k} = \lambda_k\alpha\lambda_k^{-1}$ as defined in 1.11. Define $\alpha'$ by $\alpha'(k) = k+1$, $\alpha'(k+1) = k$, $\alpha'(n) = \lambda_k^{-1}\beta\lambda_k(n)$ for other $n \in J_N$. Then

$$\beta_{\alpha',k} = \lambda_k(\lambda_k^{-1}\beta\lambda_k)\lambda_k^{-1} = \beta.$$

This $\alpha'$ is clearly unique, since $\lambda_k^{-1}$ and $\lambda_k$ are one-to-one, and since we must define $\alpha'$ so that it has a short line at $k$ if we wish $\beta_{\alpha',k} = \beta$.

To see that $\alpha \times \alpha'$ is an $N$-fold, we reverse the argument above: if

$$\beta_{\alpha,k} \times \beta = (k_1,..., k_r)(\lambda_k\alpha(k), \lambda_k\alpha(k+1))(k_{r+1},..., k_{N-2}),$$

then

$$\alpha \times \alpha' = (\lambda_k^{-1}(k_1),..., \lambda_k^{-1}(k_r))(\alpha(k), k, k+1, \alpha(k+1))(\lambda_k^{-1}(k_{r+1}),..., \lambda_k^{-1}(k_{N-2})),$$

an $N$-cycle. A second case is similar.

See Figure 5a for an illustration of the theorem.

1.17. Theorem. *Let $\alpha \times \alpha'$ be an oriented $N$-fold. If $\alpha'$ has a short line at* 1, *then $\gamma_\alpha \times \gamma_{\alpha'}$ is an oriented $(N-2)$-fold. If $\alpha'$ has a short line at $N$, then $\delta_\alpha \times \delta_{\alpha'}$ is an oriented $(N-2)$-fold. Conversely, if $\gamma_\alpha \times \gamma$ is an oriented $(N-2)$-fold, there corresponds a unique oriented $N$-fold $\alpha \times \alpha'$, where $\alpha'$ has a short line at* 1 *and $\gamma = \gamma_{\alpha'}$. If $\delta_\alpha \times \delta$ is an oriented $(N-2)$-fold, there corresponds a unique oriented $N$-fold $\alpha \times \alpha'$, where $\alpha'$ has a short line at $N$ and $\delta = \delta_{\alpha'}$.*

The proof is similar to that of the preceding theorem. See figure 5b, c for illustrations of $\gamma_\alpha \times \gamma_{\alpha'}$ and $\delta_\alpha \times \delta_{\alpha'}$.

## 2. Solving the Problem

2.1. Definition. Let $\alpha$ be an oriented $N$-pattern; $1 \leqslant k \leqslant N$; $m \geqslant 0$.

(a) Let $f(\alpha, k, m)$ be the number of oriented $N$-folds $\alpha \times \alpha'$ such that $\alpha'$ has $m$ mid lines, including one at $k$.

(b) Of these $f(\alpha, k, m)$ $N$-folds $\alpha \times \alpha'$, let $g(\alpha, k, m)$ be the number for which $\alpha'$ also has a short line at 1 and $h(\alpha, k, m)$ the number for which $\alpha'$ also has a short line at $N$.

(c) Let $f(\alpha, m)$ be the total number of different $N$-folds $\alpha \times \alpha'$ for which $\alpha'$ has $m$ mid lines.

(d) Of these $f(\alpha, m)$, let $g(\alpha, m)$ enumerate those having a short line at 1, and $h(\alpha, m)$ enumerate those having a short line at $N$.

2.2. Lemma. *Let $\theta$ be the one oriented 3-pattern. Then*

$$f(\theta, k, m) = g(\theta, k, m) = h(\theta, k, m) = 0,$$
$$f(\theta, 0) = 2, \qquad g(\theta, 0) = 1, \qquad h(\theta, 0) = 1,$$
$$f(\theta, m) = g(\theta, m) = h(\theta, m) = 0, \qquad \text{if} \quad m > 0, \quad 1 \leqslant k \leqslant N.$$

*Let $\alpha$ be an oriented $N$-pattern, $N \geqslant 5$, $1 \leqslant k \leqslant N$, $m \geqslant 0$. Then*

(1) $f(\alpha, k, m) = 0$ *if* $m = 0$ *or* $m > (N-3)/2$,
*if $\alpha$ has a mid line at $k$,*
*if* $k = 1$ *or* $k = N$.

$$f(\alpha, k, m) = f(\beta_{\alpha,k}, k-1, m) + f(\beta_{\alpha,k}, m-1) - f(\beta_{\alpha,k}, k-1, m-1)$$
*otherwise.*

(2) $g(\alpha, k, m) = 0$ *if* $k = 2$,
*if* $f(\alpha, k, m) = 0$.
$h(\alpha, k, m) = 0$ *if* $k = N-1$,
*if* $f(\alpha, k, m) = 0$.

$$g(\alpha, k, m) = g(\beta_{\alpha,k}, k-1, m) + g(\beta_{\alpha,k}, m-1) - g(\beta_{\alpha,k}, k-1, m-1)$$
$$h(\alpha, k, m) = h(\beta_{\alpha,k}, k-1, m) + h(\beta_{\alpha,k}, m-1) - h(\beta_{\alpha,k}, k-1, m-1)$$
*otherwise.*

(3) $f(\alpha, m) = \dfrac{1}{m} \sum_k f(\alpha, k, m)$ *if* $m > 0$,

$f(\alpha, 0) = g(\alpha, 0) + h(\alpha, 0)$.

(4) $g(\alpha, m) = \dfrac{1}{m} \sum_k g(\alpha, k, m)$ *if* $m > 0$,

$h(\alpha, m) = \dfrac{1}{m} \sum_k h(\alpha, k, m)$ *if* $m > 0$.

(5) $g(\alpha, 0) = \sum_m f(\gamma_\alpha, m) - \sum_{m>0} g(\alpha, m)$

$h(\alpha, 0) = \sum_m f(\delta_\alpha, m) - \sum_{m>0} h(\alpha, m)$.

Proof: It is clear by inspection of the two possible oriented 3-folds that the lemma is true if $N = 3$. Let $N$ be odd, $N \geqslant 5$; assume the lemma holds for $N-2$. Since $\beta_{\alpha,k}$, $\gamma_\alpha$, $\delta_\alpha$ are all oriented $(N-2)$-patterns, it is possible to carry out the inductive process of the lemma for each $\alpha$, by evaluating for each $k$ and $m$ in this order: $f(\alpha, k, m)$, $g(\alpha, k, m)$, $h(\alpha, k, m)$, $g(\alpha, m)$, $h(\alpha, m)$, $f(\alpha, m)$.

2.3. PROOF OF (1). Clearly $f(\alpha, k, 0) = 0$, for any $\alpha'$ must have a mid line at $k$ to be counted, so that $m \geqslant 1$ for any such $\alpha'$. Also, $f(\alpha, k, m) = 0$ for $m > (N-3)/2$, since a non-oriented $N$-pattern can have at most $(N-3)/2$ mid lines. If $\alpha$ has a short line at $k$, then $\alpha'$ cannot also have a short line at $k$, because then $\alpha \times \alpha'$ would not be an $N$-cycle. $f(\alpha, 1, m) = f(\alpha, N, m) = 0$ by definition of a mid line.

Suppose $1 \leqslant m \leqslant (N-3)/2$ and that $\alpha$ has a gap at $k$, $2 \leqslant k \leqslant N-1$. Then $\beta_{\alpha,k} \times \beta_{\alpha',k}$ is an $(N-2)$-fold and has a mid line at $k-1$ if and only if $\alpha'(k-1) \equiv k+2 \pmod{N-2}$ by Lemma 1.13. All other short lines are preserved (though possibly shifted.) By Theorem 1.16, there is a one-to-one correspondence $\alpha' \leftrightarrow \beta_{\alpha',k}$. Thus, as we let $\alpha'$ run through all the $N$-patterns having $m$ mid lines, we obtain all the $f(\beta_{\alpha,k}, k-1, m)$ $(N-2)$-folds $\beta_{\alpha,k} \times \beta$ such that $\beta$ has $m$ mid lines including one at $k-1$. We will also obtain all the $f(\beta_{\alpha,k}, m-1) - f(\beta_{\alpha,k}, k-1, m-1)$ $(N-2)$-folds $\beta_{\alpha,k} \times \beta$ such that $\beta$ has $m-1$ mid lines, but no mid line at $k-1$. This establishes (1).

2.4. PROOF OF (3): $f(\alpha, 0) = g(\alpha, 0) + h(\alpha, 0)$ since every $\alpha'$ has at least one short line, which must be at either 1 or $N$ if $\alpha'$ has no mid lines. $f(\alpha, m) = 0$ if $m > (N-3)/2$.

If $\alpha'$ has $m$ mid lines, $1 \leqslant m \leqslant (N-3)/2$, then for each $k$ at which $\alpha'$ has a mid line, the same $N$-fold $\alpha \times \alpha'$ will be counted in the evaluation $f(\alpha, k, m)$. Hence $\sum_k f(\alpha, k, m)$ is $m$ times the number of $N$-folds $\alpha \times \alpha'$ for which $\alpha'$ has $m$ mid lines.

2.5. PROOF OF (2) AND (4): Clearly $g(\alpha, k, m) = 0 = h(\alpha, k, m)$ when $f(\alpha, k, m) = 0$. If $\alpha'$ has a short line at 1, then $\alpha'$ cannot also have a short line at 2, since $\alpha'(2) = 1 \neq 3$. Thus $g(\alpha, 2, m) = 0$. Similarly, $h(\alpha, N-1, m) = 0$.

If $\alpha'$ has a short line at 1, then $\beta_{\alpha',k}$ has a short line at 1. If $\alpha'$ has a short line at $N$, then $\beta_{\alpha',k}$ has a short line at $N-2$; this is not a mid line, since $\beta_{\alpha',k}$ is an $(N-2)$-pattern.

The rest of the argument is the same as that used in 2.3 and 2.4. It is interesting to note that it is only because $g(\alpha, 2, m) = 0 = h(\alpha, N-1, m)$ that these evaluations $g$ and $h$ eventually grow much smaller than $f$.

2.6. COROLLARY. $\sum_m f(\alpha, m)$ *gives the total number of oriented*

*N-folds $\alpha \times \alpha'$. Of these, $\sum_m g(\alpha, m)$ have a short line at 1 and $\sum_m h(\alpha, m)$ have a short line at N.*

2.7. LEMMA. *Let $\alpha$ be an oriented N-pattern. Then*

$$\sum_m f(\gamma_\alpha, m) = \sum_m g(\alpha, m); \qquad \sum_m f(\delta_\alpha, m) = \sum_m h(\alpha, m).$$

*Thus (5) holds.*

PROOF: Since $\gamma_\alpha$ is an oriented $(N-2)$-pattern, we may use the induction hypothesis to see that $\sum_m f(\gamma_\alpha, m)$ gives the total number of oriented $(N-2)$-folds of the form $\gamma_\alpha \times \gamma$. To each such $\gamma_\alpha \times \gamma$ there corresponds one and only one oriented $N$-fold $\alpha \times \alpha'$, where $\alpha'$ has a short line at 1 and $\gamma = \gamma\alpha'$ by Theorem 1.17. Thus $\sum_m f(\gamma_\alpha, m)$ gives the total number of oriented $N$-folds $\alpha \times \alpha'$ such that $\alpha'$ has a short line at 1.

In a similar way, $\sum_m f(\delta_\alpha, m)$ is seen to give the total number of oriented $N$-folds $\alpha \times \alpha'$ such that $\alpha'$ has a short line at $N$.

Now apply Corollary 2.6; (5) follows at once.

Actually, $g(\alpha, 0)$ and $h(\alpha, 0)$ are either 0 or 1, since there is one $N$-pattern $\alpha'$ having a short line only at 1, and one having a short line at $N$. However, this observation is not necessary for the theory. It does mean that experimental determination of this value might be simpler for purposes of computation.

2.8. THEOREM. $P_0(N) = \sum_{\alpha,m} f(\alpha, m)$, $\quad P_0(N-1) = \sum_{\alpha,m} g(\alpha, m)$, *where the sum is taken over all the oriented N-patterns $\alpha$.*

PROOF: The formula for $P_0(N)$ is obvious. The formula for $P_0(N-1)$ is also obvious from Theorem 1.6. For the oriented $N$-folds $\pi$ such that $\pi(2) = 2$ are precisely those oriented $N$-folds $\alpha \times \alpha'$ such that $\alpha'$ has a short line at 1.

By using the formulas of Theorem 2.6, we can now find $P(N)$ for every integer $N$. See the table at the end of the paper for $P(N)$ where $N \leqslant 16$. The computer program used is available on request.

## 3. OTHER FOLDING PROBLEMS

3.1. INTRODUCTION. In the first paragraphs of this paper, we restricted our attention to the case of a strip of stamps which satisfied the properties: (1) the front side of the strip can be distinguished from the back side; (2) the left edge of the strip can be distinguished from the right edge;

(3) the top edge can be distinguished from the bottom edge. Since the problem is two dimensional, any two of these conditions imply the third. We shall now consider the folding problem when only one or none of the conditions holds.

Let us assume for the purpose of notation that the strip has been labeled so that it now satisfies (1), (2), and (3).

3.2. DEFINITION. Let $N$ be fixed; define $\rho$ by $\rho(k) = N + 1 - k$, $k \in J_N$. An $N$-fold $\pi$ is said to be symmetric if $\pi\rho(k) = \rho\pi(k)$ for all $k \in J_N$. Otherwise, $\pi$ is said to be anti-symmetric.

Let $S(N)$ denote the number of symmetric folds of a strip of $N$ stamps.

If $\pi$ is any $N$-fold, then $\rho\pi$, $\pi\rho$, and $\rho\pi\rho$ are also $N$-folds. $\rho\pi$ reverses the order of the stamps in the pile. Note that $\rho$ satisfies the definition of an $N$-pattern. Hence $\rho = \rho^{-1}$, implying $\rho\pi = \pi\rho$ if and only if $\pi = \rho\pi\rho$.

3.3. EXAMPLE. The fold $\pi = 2, 1, 4, 3$ is symmetric since $\rho\pi\rho = \pi$, $\rho\pi = 3, 4, 1, 2 = \pi\rho$. The fold $\pi = 1, 3, 4, 2$ is anti-symmetric since $\rho\pi = 4, 2, 1, 3$; $\pi\rho = 2, 4, 3, 1$; $\rho\pi\rho = 3, 1, 2, 4$ (see Fig. 2).

If the strip is blank, that is, if it satisfies none of conditions (1), (2), and (3), then each $N$-fold $\pi$ will be indistinguishable from $\rho\pi$, $\pi\rho$, and $\rho\pi\rho$, and will be distinguishable from all other $N$-folds. $\pi$ and $\rho\pi\rho$, $\rho\pi$ and $\pi\rho$ will be identical if $\pi$ is symmetric.

If the strip satisfies exactly one of the three conditions, then each $N$-fold $\pi$ will be distinguishable from all other folds except for one corresponding fold which is indistinguishable from $\pi$. This corresponding fold will be identical to $\pi$ if $\pi$ is symmetric and the fold is $\rho\pi\rho$. If condition (1) alone holds, for example, if the strip is blank, with one side red and the other black, then the corresponding fold to $\pi$ is $\rho\pi\rho$ if $N$ is even, and $\rho\pi$ if $N$ is odd. If condition (2) alone holds, for example, if the strip is entirely blank except for one stamp on one end which is red on both sides, then the corresponding fold to $\pi$ is $\pi\rho$. If condition (3) alone holds, for example, if the strip is blank on both sides and only one of the long edges is perforated, then the corresponding fold to $\pi$ is $\rho\pi$ if $N$ is even, and $\rho\pi\rho$ if $N$ is odd.

We leave it to the reader to see that these statements describe the physical situation. We formulate them into a theorem:

3.4. THEOREM. *Let $B(N)$ be the number of different folds of a strip of $N$ stamps which satisfy none of the conditions listed in* 3.1. *Then*

$$B(N) = \tfrac{1}{4}(P(N) - S(N)) + \tfrac{1}{2}S(N). \quad = \tfrac{1}{4}(P(N) + S(N))$$

*Let $F(N)$ be the number of different folds of a strip of stamps satisfying exactly one of the conditions of* 3.1.

*Then*

$F(N) = 2B(N)$ *if N is even and condition* (1) *is satisfied, or if N is odd and condition* (3) *is satisfied;*

$F(N) = \frac{1}{2}P(N)$ *in all other cases.*

We now proceed to find $S(N)$ for each $N$. We shall first consider the case when $N$ is even.

3.5. Lemma. *Let $N = 2M$. Let $\pi$ be a symmetric $N$-fold. Then $\pi(1), \pi(2),..., \pi(M)$ are mapped onto either $J_M$ or $J_N - J_M$.*

Proof: The lemma is clearly true if $N = 2$. Let $N > 2$; assume the lemma is true for all symmetric $(N - 2)$-folds. Suppose $\pi(2) = k \leqslant M$. Then we may apply the induction hypothesis to the $(N - 2)$-subfold $\pi(2),..., \pi(N - 1)$ to see that $\pi(2),..., \pi(M)$ maps into $J_M$.

Let $\pi(1) = j$. Suppose $j \geqslant M + 1$. If $j > N + 1 - k = \rho(k) = \rho\pi(2) = \pi\rho(2) = \pi(N - 1)$, then $\pi(N) = N + 1 - j < k$. Thus $\pi(N) \to \pi(2) \to \pi(N-1) \to \pi(1)$, contradicting 1.4(2). If $M < j < N + 1 - k$, then $\pi(2) \to \pi(N) \to \pi(1) \to \pi(N - 1)$, another contradiction. Hence $\pi(1) \leqslant M$ as desired.

Similarly, if $\pi(2) > M$, then $\pi$ maps $J_M$ onto $J_N - J_M$.

From this lemma, we may look upon symmetric $2M$-folds as a combination of two $M$-folds, one of which is a reverse copy of the other, and one of which contains 1. (See Figure 6a.) Thus the first stamp of one of these folds must be able to be joined to the last stamp of the other. One of these M-folds completely determines the other by the definition of symmetric folds. These considerations yield:

3.6. Lemma. *To each $M$-fold $\pi$ there correspond two unique symmetric $2M$-folds if and only if $\pi$ extends simply (where by "extending simply" we mean that $\pi(1), \pi(2),..., \pi(M), M + 1$ is also an $(M + 1)$-fold, denoted by $\pi^+$; see Figure* 6a).

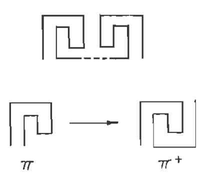


Figure 6a

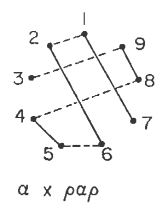


Figure 6b

3.7. Theorem. $S(2M) = 2P_0(M + 1)$.

Proof: Let $\tau$ be an oriented $(M + 1)$-fold. $\rho\tau\rho$ is also an $(M + 1)$-fold and $\rho\tau\rho(M+1) = \rho\tau(1) = \rho(1) = M+1$. Hence $\rho\tau\rho = \pi^+$ is a simple extension of the $M$-fold $\pi$ determined by $\rho\tau\rho$ restricted to $J_M$.

Conversely, if $\pi$ extends simply to $\pi^+$, then $\pi^+$ is an oriented $(M+1)$-fold that is uniquely determined by $\pi$.

3.8. THEOREM. *Let $\pi = \alpha \times \alpha'$ be an $N$-fold, where $N = 2M - 1 \geqslant 3$. Then*

(1) $\alpha' = \rho\alpha\rho \Rightarrow$ (2) $\pi$ is symmetric;

(2) $\Rightarrow$ (3) $\alpha \times \rho$ is an $N$-fold;

(3) $\Rightarrow$ (4) $\alpha \times \rho\alpha\rho$ is an $N$-fold.

PROOF: A more precise proof would employ several inductive arguments; however, the essential steps are given in the shorter treatment below. See Figure 6b for an example.

(1) $\Rightarrow$ (2). If $\alpha' = \rho\alpha\rho$, then

$$\rho\pi\rho = \rho\alpha\rho \times \rho\alpha'\rho = \rho\alpha\rho \times \rho(\rho\alpha\rho)\rho = \rho\alpha\rho \times \alpha = \alpha \times \rho\alpha\rho = \alpha \times \alpha' = \pi.$$

Hence $\pi$ is symmetric.

(2) $\Rightarrow$ (3). 
$$\begin{aligned}\pi &= (\pi(1), \pi(2), \pi(3),\ldots, \pi(N))\\ &= \pi(1)(\pi(1), \pi(2))(\pi(2), \pi(3)),\ldots, (\pi(N-1), \pi(N))\pi(N)\\ &= \pi(1)(\pi(1), \alpha'\pi(1))(\pi(2), \alpha\pi(2)),\ldots,(\pi(N-1), \alpha\pi(N-1))\pi(N),\end{aligned}$$

where
$$\alpha(\pi(1)) = \pi(1), \qquad \alpha'(\pi(N)) = \pi(N),$$
$$\alpha(\pi(2s)) = \pi(2s+1), \qquad \alpha'(\pi(2s)) = \pi(2s-1),$$
$$\alpha(\alpha(k)) = k = \alpha'(\alpha'(k)).$$

If $\pi$ is symmetric, then $\rho\pi = \pi\rho$, and we have:

$$\begin{aligned}\alpha \times \rho &= \pi(1)(\pi(1), \rho\pi(1))(\pi(2), \alpha\pi(2)),\ldots\\ &= \pi(1)(\pi(1), \pi\rho(1))(\pi\rho(1), \alpha\pi\rho(1)),\ldots\\ &= \pi(1)(\pi(1), \pi(N))(\pi(N), \pi(N-1))(\pi(N-1), \pi(2))(\pi(2), \pi(3)),\ldots,\end{aligned}$$

an $N$-cycle.

(3) $\Rightarrow$ (4). Suppose $\alpha(k) = k$. Then

$$\begin{aligned}\alpha \times \rho &= \alpha(k)(\alpha(k), \rho\alpha(k))(\rho\alpha(k), \alpha\rho\alpha(k),\ldots \text{ and}\\ \alpha \times \rho\alpha\rho &= \alpha(k)(\alpha(k), \rho\alpha\rho\alpha(k))(\rho\alpha\rho\alpha(k), \alpha\rho\alpha\rho\alpha(k)),\ldots\end{aligned}$$

Define a function $f$ inductively by

$$f(1) = \alpha(k), \qquad f(2s) = \rho f(2s-1), \qquad f(2s+1) = \alpha f(2s).$$

Then we may write

$$\alpha \times \rho = f(1)(f(1), f(2))(f(2), f(3)),\ldots, (f(N-1), f(N))f(N).$$

Note that $f(N) = \rho f(N) = M$, since an $N$-cycle $\alpha \times \rho$ must end at $M$, the fixed point of $\rho$. This relationship implies that $f(N + 1) = \rho f(N) = f(N) = f(N + 1 - 1), f(N + 2) = \alpha f(N + 1) = \alpha f(N) = \alpha\alpha f(N - 1) = f(N - 1) = f(N + 1 - 2)$, and, in general, $f(N + k) = f(N + 1 - k)$, $k \in J_N$. Two situations occur for $\alpha \times \rho\alpha\rho$:

(I) If $N \equiv 1 \pmod 4$, then

$$\begin{aligned}\alpha \times \rho\alpha\rho &= (f(1), f(4), f(5), \ldots, f(N - 1), f(N), f(N + 3), f(N + 4), \ldots)\\ &= (f(1), f(4), f(5), \ldots, f(N-1), f(N), f(N-2), f(N-3), \ldots, f(3), f(2)).\end{aligned}$$

(II) If $N \equiv 3 \pmod 4$, then

$$\begin{aligned}\alpha \times \rho\alpha\rho &= (f(1), f(4), f(5), \ldots, f(N-3), f(N-2), f(N + 1), f(N + 2), \ldots)\\ &= (f(1), f(4), f(5), \ldots, f(N-3), f(N-2), f(N), f(N-1), \ldots, f(3), f(2)).\end{aligned}$$

In either case, $\alpha \times \rho\alpha\rho$ is an $N$-cycle.

Let $f(\alpha, m)$ be defined as in 2.1(c).

3.9. Corollary. $S(2M - 1) = \sum_m f(\sigma_M \rho \sigma_M^{-1}, m)$.

Proof: $\alpha \times \rho$ is an $N$-fold $\Leftrightarrow \rho \times \alpha$ is an $N$-fold $\Leftrightarrow \sigma_M \rho \sigma_M^{-1} \times \sigma_M \alpha \sigma_M^{-1}$ is an oriented $N$-fold. As we let $\alpha$ run through all $N$-patterns, so does $\sigma_M \alpha \sigma_M^{-1}$, and we obtain all the $\sum_m f(\sigma_M \rho \sigma_M^{-1}, m)$ oriented $N$-folds $\sigma_M \rho \sigma_M^{-1} \times \alpha'$.

unlabeled

4. Table of Results for $N \leqslant 16$

| $N$ | $P_0(N)$ | $P(N)$ | $S(N)$ | $B(N)$ |
|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 |
| 2 | 1 | 2 | 2 | 1 |
| 3 | 2 | 6 | 2 | 2 |
| 4 | 4 | 16 | 4 | 5 |
| 5 | 10 | 50 | 6 | 14 |
| 6 | 24 | 144 | 8 | ~~39~~ 38 |
| 7 | 66 | 462 | 18 | 120 |
| 8 | 174 | 1392 | 20 | ~~358~~ 353 |
| 9 | 504 | 4536 | 56 | ~~1176~~ 1148 |
| 10 | 1406 | 14060 | 48 | 3527 ✓ |
| 11 | 4210 | 46310 | 178 | 11622 ✓ |
| 12 | 12198 | 146376 | 132 | 36627 ✓ |
| 13 | 37378 | 485914 | 574 | 121622 ✓ |
| 14 | 111278 | 1557892 | 348 | 389560 ✓ |
| 15 | 346846 | 5202690 | 1870 | 1301140 ✓ |
| 16 | 1053874 | 16861984 | 1008 | 4215748 ✓ |

682 136 1010 1011

and 3054 (the bad version)

want them for N ≤ 22

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