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Dear Prof. Shapiro,

Thanks for sending the reprint of your paper with Donaghey that surveys the Motzkin numbers \([JCT A23 (1977), 291–301]\).

I happened to notice that the sequence \(\gamma_n\) you mention on page 293 is precisely half of the sequence that Euler called a "memorable failure of induction" in 1765 (see the answer to exercise 7.56 in *Concrete Mathematics*, and reference 91 in the bibliography): The numbers are \(F_{n-1}(F_{n-1} + 1)/2\) for \(n = 0, 1, 2, 3, \ldots, 8\), but then the pattern stops!

I found this by noticing that \((1 - x)^2 - 4x^2 = (1 - 3x)(1 + x)\), hence the generating function in your equation (7) can be divided by \(1 + x\) and you still get essentially a polynomial multiple of the generating function for the \(\beta\)'s. Indeed,

\[
\gamma_n = \frac{1}{2}(3\beta_n - \beta_{n+1}).
\]

From this, or from your equation for \(M_n\) at the bottom of page 293,

\[
m_n = \frac{1}{2}(3\beta_n + 2\beta_{n+1} - \beta_{n+2}).
\]

Thus Euler's numbers \(\beta_n\) give a nice "basis" for both the Motzkin numbers and their \(\gamma\) relations. Since the generating function for \(\beta_n\) is

\[
\frac{1}{\sqrt{(1 - 3x)(1 + x)}} = \frac{\sqrt{3}}{2} \frac{1}{\sqrt{1 - 3x}} \frac{1}{\sqrt{1 - (1 - 3x)/4}} = \sum_{n=0}^{\infty} x^n \sum_{k=0}^{\infty} \binom{k - 1/2}{n} \frac{\sqrt{3}}{2} \left(\frac{-1}{4}\right)^k (-3)^n
\]

\[
= \sum_{n=0}^{\infty} x^n \frac{3^{n+1/2}}{2} \sum_{k=0}^{\infty} \binom{n - k - 1/2}{n} \left(-\frac{1}{4}\right)^k
\]

\[
= \sum_{n=0}^{\infty} x^n \frac{3^{n+1/2}}{\sqrt{4\pi}} \sum_{k=0}^{\infty} \frac{k!}{(2k)!} n^{k+1/2}
\]

we have the asymptotic formula

\[
\beta_n \approx 3^n \sqrt{\frac{3}{4\pi n}} \left(1 + \frac{5}{8n} + O(n^{-2})\right).
\]
(See the answers to exercises 9.44 and 9.60.) This gives asymptotics for $m_n$ and $\gamma_n$.

Cordially,

Donald E. Knuth
Professor