Signum Equations and Extremal Coefficients

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Let $a(n)$ denote the number of sign choices $+$ and $-$ such that

$$\pm 1 \pm 2 \pm 3 \pm \cdots \pm n = 0$$

and $b(n)$ denote the number of solutions of

$$\varepsilon_1 \cdot 1 + \varepsilon_2 \cdot 2 + \varepsilon_3 \cdot 3 + \cdots + \varepsilon_n \cdot n = 0$$

where each $\varepsilon_j \in \{-1, 0, 1\}$. It can be proved that $[1, 2]$

- $a(n)$ is the coefficient of $x^{n(n+1)/2}$ in the polynomial $\prod_{k=1}^{n} (1 + x^{2k})$,
- $b(n)$ is the coefficient of $x^{n(n+1)/2}$ in the polynomial $\prod_{k=1}^{n} (1 + x^k + x^{2k})$.

Clearly $a(n) = 0$ when $n \equiv 1, 2 \pmod{4}$. If we think of sign choices as independent random variables with equal weight on $\{-1, 1\}$, then

$$E \left( \sum_{k=1}^{n} \pm k \right) = 0, \quad \text{Var} \left( \sum_{k=1}^{n} \pm k \right) = \frac{n(n+1)(2n+1)}{6} \sim \frac{n^3}{3}$$

as $n \to \infty$. By the Central Limit Theorem,

$$P \left( \sqrt{3n^{-3/2}} \sum_{k=1}^{n} \pm k \leq x \right) \sim \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp \left( -\frac{t^2}{2} \right) dt$$

which implies that $[3, 4]$

$$P \left( \sum_{k=1}^{n} \pm k = 0 \right) \sim s \sqrt{\frac{3}{2\pi}} n^{-3/2} \exp \left( -\frac{x^2}{2} \right) \bigg|_{x=0}$$

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where $s = 1 - (-1) = 2$ is the span of the distribution of $\pm$; hence \[5, 6\]
\[
a(n) \sim \sqrt{\frac{6}{\pi}} n^{-3/2} 2^n.
\]
In the same way,
\[
b(n) \sim \frac{1}{2\sqrt{\pi}} n^{-3/2} 3^{n+1}.
\]
Let $c(n)$ denote the number of sign choices such that
\[
\pm 1 \pm 2 \pm 3 \pm \cdots \pm n = \pm 1 \pm 2 \pm 3 \pm \cdots \pm n.
\]
Here \[7\]
\[c(n) \text{ is the coefficient of } x^{(n+1)/2} \text{ in the polynomial } \prod_{k=1}^{n} (1 + x^k)^2\]
and \[8, 9, 10, 11\]
\[c(n) \sim \sqrt{\frac{3}{\pi}} n^{-3/2} 2^n.
\]
Define \[12\]
\[
\alpha(n) \text{ to be the maximal coefficient in the polynomial } \prod_{k=1}^{n} (1 + x^{2k}),
\]
\[
\beta(n) \text{ to be the maximal coefficient in the polynomial } \prod_{k=1}^{n} (1 + x^k + x^{2k}),
\]
\[
\gamma(n) \text{ to be the maximal coefficient in the polynomial } \prod_{k=1}^{n} (1 + x^k)^2.
\]
The first of these has an immediate combinatorial interpretation: $\alpha(n)$ is the number of sign choices such that
\[
\pm 1 \pm 2 \pm 3 \pm \cdots \pm n \text{ is 0 or 1.}
\]
While $\beta(n)$ seems not to have such a representation, the last sequence satisfies trivially $\gamma(n) = c(n)$ always.

We look at several more examples. Define \[13\]
\[
\lambda_{\max}(n) \text{ to be the maximal coefficient in } \prod_{k=1}^{n} (1 - x^{2k})
\]
and $-\lambda_{\min}(n)$ to be the corresponding minimal coefficient;
\(\mu_{\text{max}}(n)\) to be the maximal coefficient in \((-1)^n \prod_{k=1}^{n} (1 - x^k)^2\)

and \(- \mu_{\text{min}}(n)\) to be the corresponding minimal coefficient.

Only the third of these possesses a clear simplification:

\[\mu_{\text{max}}(n)\] is the coefficient of \(x^{n(n+1)/2}\) in \((-1)^n \prod_{k=1}^{n} (1 - x^k)^2\)

and the asymptotics

\[\mu_{\text{max}}(n)^{1/n} \sim 1.48... \sim 2 e^{-0.29...}\]

are of interest [14, 15]. Greater understanding of the other sequences is desired.

0.1. Number Partitioning. What is the number of ways to partition the set \(\{1, 2, \ldots, n\}\) into two subsets whose sums are as nearly equal as possible? If \(n \equiv 0, 3 \text{ mod } 4\), the answer is \(\alpha(n)\); if \(n \equiv 1, 2 \text{ mod } 4\), the answer is \(\alpha(n)/2\). In the former case, the subsets have the same sum; in the latter, the subsets have sums that differ by 1 [16, 17]. Partitioning arbitrary sets of \(n\) integers, each typically of order \(2^n\), is an NP-complete problem. The ratio \(m/n\) characterizes the difficulty in searching for a perfect partition (one in which subset sums differ by at most 1). A phase transition exists for this problem (at \(m/n = 1\), in fact) and perhaps similarly for all NP problems [17, 18, 19].

As an aside, we observe that

\[\lambda_{\text{max}}(n)\] is the coefficient of \(x^{n(n+1)/2}\) in the polynomial \(\prod_{k=1}^{n} (1 - x^{2k})\)

for \(n \equiv 0 \text{ mod } 4\), but this fails elsewhere (a conjectural relation involving \(x^{(n+1)^2/2}\) coefficients for \(n \equiv 3 \text{ mod } 4\) fails apart when \(n = 27\)). It seems to be true that

\[\lambda_{\text{max}}(n)^{1/n} \sim 1.21... \sim 2 e^{-0.50...}\]

as \(n \to \infty\) via multiples of 4.

As another aside, if \(d(n)\) is the number of solutions of

\[\varepsilon_1 \cdot 1 + \varepsilon_2 \cdot 2 + \varepsilon_3 \cdot 3 + \cdots + \varepsilon_n \cdot n = \varepsilon_{-1} \cdot 1 + \varepsilon_{-2} \cdot 2 + \varepsilon_{-3} \cdot 3 + \cdots + \varepsilon_{-n} \cdot n,\]

then [20]

\(d(n)\) is the coefficient of \(x^{n(n+1)}\) in the polynomial \(\prod_{k=1}^{n} (1 + x^k + x^{2k})^2\)

(in fact, it is the maximal such coefficient)
and
\[ d(n) \sim \frac{1}{2\sqrt{2\pi}} n^{-3/2} 3^{2n+1}. \]

This grows more quickly than \( b(n) \), of course. We wonder what else can be said in both cases. For example, what is the mean percentage of 0s in \( \{e_j\} \) taken over all solutions, as \( n \to \infty \)? It may well be 1/3 for both, but it may be \( > 1/3 \) for one or the other.

**0.2. Addendum.** Define a function \( G : (0, 1) \to \mathbb{R} \) by
\[ G(x) = \int_0^1 \ln (\sin(\pi xt)) \, dt. \]

There is a unique point \( x_0 = 0.7912265710... \) at which \( G \) attains its maximum value \( G(x_0) = -0.4945295653... \). Let
\[ r = \exp(2G(x_0)) = 0.3719264606... = \frac{1}{4} (1.4877058426...), \]
\[ C = \frac{4\sin(\pi x_0)}{x_0} \sqrt{-G''(x_0)} = 2.4057458393... \]

then [21]
\[ \mu_{\max}(n) \sim C \left(\frac{4r}{\sqrt{n}}\right)^n \]
as \( n \to \infty \), making impressively precise our earlier conjecture. An analogous formula for \( \lambda_{\max}(n) \) for \( n \equiv 0 \mod 4 \) remains open.

**References**


