Pi in Golden Ratio Base Phi

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Abstract

π and φ (Golden Ratio) have fascinated mathematicians, scientists and artists for many centuries. Both are irrational numbers, but still related to integers in many ways. This article proves a new BBP-like formula for π in base φ.

1 Formula

\[
\pi = \frac{4}{\phi} + \sum_{n=0}^{\infty} \frac{1}{\phi^{12n}} \left[ \frac{8\phi^{-3}}{12n+3} + \frac{4\phi^{-5}}{12n+5} - \frac{4\phi^{-7}}{12n+7} - \frac{8\phi^{-9}}{12n+9} - \frac{4\phi^{-11}}{12n+11} + \frac{4\phi^{-13}}{12n+13} \right]
\]  

(1)

where

\[
\phi = \frac{1 + \sqrt{5}}{2}
\]  

(2)

The equation 1 gives value of π sequentially from left to right providing increasing precision of 12 fractional places in base φ with each iteration.

2 Identity

equation 1 is derived from the following geometric relation between π and φ

\[
\frac{\pi}{4} = \arctan \left( \frac{1}{\phi} \right) + \arctan \left( \frac{1}{\phi^3} \right)
\]  

(3)

We will first prove the equation 3 and then derive equation 1
3 Proof

Formula for addition of tan is

$$\tan (A + B) = \frac{\tan (A) + \tan (B)}{1 - \tan (A) \tan (B)}$$

(4)

tan of right-hand side of equation 3 gives

$$= \frac{\frac{1}{\phi} + \frac{1}{\phi^2}}{1 - \frac{1}{\phi} \frac{1}{\phi^2}}$$

(5)

simplifying further

$$= \frac{\phi^3 + \phi}{\phi^4 - 1}$$

(6)

simplifying further

$$= \frac{\phi(\phi^2 + 1)}{(\phi^2 - 1)(\phi^2 + 1)}$$

(7)

since $\phi^2 - 1 = \phi$ hence

$$= \frac{\phi(\phi^2 + 1)}{\phi(\phi^2 + 1)}$$

(8)

simplifying further

$$= 1 = \tan \left( \frac{\pi}{4} \right)$$

(9)

which is left-hand side of equation 3. Hence proving equation 3.

4 Derivation

Taylor series for arctan ($x$) is
\[
\arctan(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)x^{2n+1}}
\]  
(10)

substituting \(\frac{1}{\phi}\) and \(\frac{1}{\phi^3}\) in equation 10 gives

\[
\arctan\left(\frac{1}{\phi}\right) = \frac{1}{\phi} - \frac{1}{3\phi^3} + \frac{1}{5\phi^5} - \frac{1}{7\phi^7} + \frac{1}{9\phi^9} - \frac{1}{11\phi^{11}} + \frac{1}{13\phi^{13}} \\
- \frac{1}{15\phi^{15}} + \frac{1}{17\phi^{17}} - \frac{1}{19\phi^{19}} + \frac{1}{21\phi^{21}} - \frac{1}{23\phi^{23}} + \frac{1}{25\phi^{25}} - \cdots
\]  
(11)

\[
\arctan\left(\frac{1}{\phi^3}\right) = \frac{1}{\phi^3} - \frac{1}{3\phi^9} + \frac{1}{5\phi^{15}} - \frac{1}{7\phi^{21}} + \frac{1}{9\phi^{27}} - \frac{1}{11\phi^{33}} + \frac{1}{13\phi^{39}} \\
- \frac{1}{15\phi^{45}} + \frac{1}{17\phi^{51}} - \frac{1}{19\phi^{57}} + \frac{1}{21\phi^{63}} - \frac{1}{23\phi^{69}} + \frac{1}{25\phi^{75}} - \cdots
\]  
(12)

adding and arranging equation 11 and equation 12 gives

\[
\frac{\pi}{4} = \frac{1}{\phi} + \frac{1}{\phi^3} - \frac{1}{3\phi^3} + \frac{1}{5\phi^5} - \frac{1}{7\phi^7} + \frac{1}{9\phi^9} - \frac{1}{11\phi^{11}} + \frac{1}{13\phi^{13}} \\
+ \left(\frac{1}{5\phi^{15}} - \frac{1}{15\phi^{15}}\right) + \frac{1}{17\phi^{17}} - \frac{1}{19\phi^{19}} - \left(\frac{1}{7\phi^{21}} - \frac{1}{21\phi^{21}}\right) - \frac{1}{23\phi^{23}} + \frac{1}{25\phi^{25}} - \cdots
\]  
(13)

combining equal \(\phi\) power terms from equation 13 gives

\[
\frac{\pi}{4} = \frac{1}{\phi} + \frac{2}{3\phi^3} + \frac{1}{5\phi^5} - \frac{2}{7\phi^7} - \frac{1}{9\phi^9} - \frac{1}{11\phi^{11}} + \frac{1}{13\phi^{13}} \\
+ \frac{2}{15\phi^{15}} + \frac{1}{17\phi^{17}} - \frac{1}{19\phi^{19}} - \frac{2}{21\phi^{21}} - \frac{1}{23\phi^{23}} + \frac{1}{25\phi^{25}} - \cdots
\]  
(14)

multiplying both sides by 4 and arranging equation 14 in groups of 6 terms gives

\[
\pi = 4\frac{1}{\phi} + \left(\frac{2}{3\phi^3} + \frac{1}{5\phi^5} - \frac{2}{7\phi^7} - \frac{1}{9\phi^9} - \frac{1}{11\phi^{11}} + \frac{1}{13\phi^{13}}\right) \\
+ \frac{1}{\phi^{12}}\left(\frac{2}{15\phi^3} + \frac{1}{17\phi^5} - \frac{1}{19\phi^7} - \frac{2}{21\phi^9} - \frac{1}{23\phi^{11}} + \frac{1}{25\phi^{13}}\right) - \cdots
\]  
(15)
which represented as series gives us equation 1.

\[
\pi = 4 \frac{1}{\phi} + \sum_{n=0}^{\infty} \frac{1}{\phi^{12n}} \left[ \frac{8\phi^{-3}}{12n + 3} + \frac{4\phi^{-5}}{12n + 5} - \frac{4\phi^{-7}}{12n + 7} - \frac{8\phi^{-9}}{12n + 9} - \frac{4\phi^{-11}}{12n + 11} + \frac{4\phi^{-13}}{12n + 13} \right]
\]

(16)

5 References

1. https://oeis.org/A102243
5. https://www.goldennumber.net/pi-phi-fibonacci/

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