A triangle for calculating the Salié numbers A000795.

Peter Bala, April 24, 2017

Oste and van der Jeugt [1, Section 7] show that a continued fraction of the form

\[
\frac{1}{1 - xd_0 - \frac{xh_1}{1 - xd_1 - \frac{xh_2}{1 - xd_2 - \frac{xh_3}{1 - xd_3 - \cdots}}}}
\]  

(1)

is the generating function for 2-Motzkin paths weighted by the integers \(d_i\) and \(h_i\). This combinatorial interpretation allows one to rapidly calculate the terms of a sequence whose generating function can be expressed as a continued fraction of the form (1). The results are conveniently displayed in the form of a lower triangular array, where the \(d\)'s occur as multiplication factors along diagonals of the array and the \(h\)'s as horizontal multiplication factors along rows of the array. In the particular case of the Salié numbers A000795, the generating function can be expressed as the continued fraction

\[
\frac{1}{(1-2x/(1-4x/(1-10x/(1-16x/(1-\cdots-(4n^2+4n+2)x/(1-4n^2x/(1-\cdots)))))))}.
\]

So in this case the \(d\)'s are all zero and the horizontal multiplication factors are given by the sequence \(a(2n) = 4n^2, a(2n + 1) = 4n^2 + 4n + 2\). The Salié number sequence is the leading diagonal of the following lower triangular array:

\[
\begin{array}{cccccccc}
1 \\
\downarrow \\
1 & \rightarrow & 2 & \\
\downarrow & \downarrow & \\
1 & \rightarrow & 6 & \rightarrow & 12 & \\
\downarrow & \downarrow & \downarrow & \downarrow & \\
1 & \rightarrow & 16 & \rightarrow & 76 & \rightarrow & 152 & \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\
1 & \rightarrow & 32 & \rightarrow & 396 & \rightarrow & 1736 & \rightarrow & 3472 & \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\
1 & \rightarrow & 58 & \rightarrow & 1324 & \rightarrow & 14976 & \rightarrow & 63376 & \rightarrow & 126752
\end{array}
\]

References