The situation becomes more complicated when $t > 2$. We are unable to give as exact results as obtained for $t = 2$, but use of the principle of inclusion and exclusion enables us to develop a method which yields the asymptotic expansion of $F(n,t)$. Here we get the probability that a random permutation of $1, 2, \cdots, n$ has no runs of length t, $t > 2$, approaches 1 as $n \to \infty$.

2. Permutations With No Runs of Length Two

It is convenient to work with the function $G(n,t) = (1/n) F(n,t)$. If we consider two permutations as equivalent, if they are cyclic shifts of each other, then $G(n,t)$ enumerates the number of equivalence classes of permutations containing no permutation with a run of length t.

We claim that the number of classes of cyclically equivalent permutations containing precisely k runs of length 2, $1 \leq k \leq n-2$, is $\binom{n}{k} G(n-k, 2)$. For there are $\binom{n}{k}$ ways of choosing which k runs occur, namely, the $\binom{n}{k}$ ways of choosing k elements from the set $\{12, 23, 34, \cdots, n1\}$. Each choice partitions the set $\{1, 2, \cdots, n\}$ into $n-k$ subsets, each subset representing symbols which must remain adjacent. For instance, the choice $12, 23, 56$ when $n = 7$ gives the partition $123, 4, 56, 7$. The number of cyclically inequivalent ways of rearranging the parts of the partition without introducing any new runs is clearly $G(n-k, 2)$, and the assertion follows.

When $k = n-1$ the argument breaks down, as the partition $123 \cdots n$ introduces the extra run $n1$. If we define $G(0,t) = 1$, then the above result is also valid for $k = n$. Hence, we obtain the recurrence relation

$$G(n,2) = (n-1)! - \left[\sum_{k=1}^{n} \binom{n}{k} G(n-k, 2) - n \right] \qquad (1)$$

The $-n$ term appears to cancel the term $\binom{n}{1} G(1,2) = n$. If we put $G_k = G(k,2)$, then Eq. (1) can be written symbolically as

$$(n-1)! + n = (1 + G)^n, \qquad (2)$$

where exponents are changed to subscripts after expanding the right side by the binomial theorem. Eq. (2) provides a rapid method for calculating $G(n,2)$. Table 2 gives the values for $1 \leq n \leq 15$.

We now define the generating functions

$$G(x) = \sum_{n=0}^{\infty} \frac{G(n,2)}{n!} x^n,$$

$$F(x) = \sum_{n=0}^{\infty} \frac{F(n,2)}{n!} x^n,$$

Table 2. Values of $G(n,2) = 1/n\, F(n,2)$

n	G (n, 2)
1	1
2	0
3	1
4	1
5	8
6	36
7	229
8	1,625
9	13,208
10	120,288
11	1,214,673
12	13,496,897
13	162,744,944
14	2,128,047,988
15	29,943,053,061

Clearly $F(x) = 1 + xG'(x)$. We compute

$$\begin{aligned} G(x)\,e^x &= \left(\sum_{n=0}^{\infty} \frac{G(n,2)}{n!} x^n \right) \left(\sum_{m=0}^{\infty} \frac{x^m}{m!} \right) \\ &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \frac{G(k,2)}{k!\,(n-k)!} \right) x^n \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \left(\sum_{k=0}^{n} \binom{n}{k} G(k,2) \right) x^n \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \left[n + (n-1)! \right] x^n \\ &= \sum_{n=1}^{\infty} \frac{x^n}{n} + x \sum_{n=0}^{\infty} \frac{x^n}{n!} + 1 \\ &= -\log(1-x) + xe^x + 1. \end{aligned}$$

Hence

$$G(x) = e^{-x} \left[1 - \log(1-x) \right] + x, \qquad (3)$$

and

$$F(x) = 1 + xG'(x) = xe^{-x} \left(\frac{x}{1-x} + \log(1-x) \right) + x + 1.$$

From Eq. (3) we get

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{G(n,2)}{n!} x^n &= \left(\sum_{i=0}^{\infty} \frac{(-1)^i x^i}{i!} \right) \left(1 + \sum_{j=1}^{\infty} \frac{x^j}{j} \right) + x \\ &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n-1} \frac{(-1)^k}{k!\,(n-k)} \right) x^n + \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n!} + x \end{aligned}$$

Equating coefficients gives

$$G(n,2) = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!\,(n-k)} + (-1)^n, \qquad n \neq 1 \tag{4}$$

and

$$F(n,2) = nG(n,2) = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!} \frac{n}{n-k} + n(-1)^n, \qquad n \neq 1. \tag{5}$$

Eqs. (4) and (5) can also be obtained from the principle of inclusion and exclusion (Ref. 3, Ch. 3 or Ref. 5, Ch. 2).

We now use Eq. (5) to obtain the asymptotic expansion of $F(n,2)/n!$. We have

$$\frac{F(n,2)}{n!} = \sum_{k=0}^{n} \frac{(-1)^k}{k!} \frac{n}{n-k} + \frac{(-1)^n}{(n-1)!}$$

$$= \sum_{k=0}^{n} \frac{(-1)^k}{k!} \left[1 + \frac{k}{n} + \left(\frac{k}{n}\right)^2 + \cdots + \left(\frac{k}{n}\right)^r + \frac{n}{n-k}\left(\frac{k}{n}\right)^{r+1} \right] + \frac{(-1)^n}{(n-1)!}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \left[1 + \frac{k}{n} + \left(\frac{k}{n}\right)^2 + \cdots + \left(\frac{k}{n}\right)^r \right] + \sum_{k=0}^{n} \frac{(-1)^k}{k!} \frac{n}{n-k} \left(\frac{k}{n}\right)^{r+1} - \sum_{k=n+1}^{\infty} \frac{(-1)^k}{k!} \left[1 + \frac{k}{n} + \left(\frac{k}{n}\right)^2 + \cdots + \left(\frac{k}{n}\right)^r \right] + \frac{(-1)^n}{(n-1)!}.$$

Hence

$$n^r \left| \frac{F(n,2)}{n!} - \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \left[1 + \frac{k}{n} + \left(\frac{k}{n}\right)^2 + \cdots + \left(\frac{k}{n}\right)^r \right] \right|$$

$$= \left| \sum_{k=0}^{n-1} \frac{(-1)^k k^{r+1}}{k!\,(n-k)} - n^r \sum_{k=n+1}^{\infty} \frac{(-1)^k}{k!} \left[\frac{1 - \left(\frac{k}{n}\right)^{r+1}}{1 - \frac{k}{n}} \right] + \frac{(-1)^n n^r}{(n-1)!} \right|$$

$$\leq \sum_{k=0}^{\log n} \frac{k^{r+1}}{k!\,(n-k)} + \sum_{k=\log n}^{n-1} \frac{k^{r+1}}{k!\,(n-k)} + n^r \left| \sum_{k=n+1}^{\infty} \frac{(-1)^k}{k!} \frac{1 - \left(\frac{k}{n}\right)^{r+1}}{1 - \frac{k}{n}} \right| + \frac{n^r}{(n-1)!}$$

$$< \frac{(\log n)(\log n)^{r+1}}{n - \log n} + \frac{n^{r+1} \cdot n}{(\log n)!} + n^r \frac{1}{(n+1)!} \left[\frac{1 - \left(\frac{n+1}{n}\right)^{r+1}}{1 - \left(\frac{n+1}{n}\right)} \right] + \frac{n^r}{(n-1)!}$$

The next to last term is obtained by using the fact that the error in truncating an alternating series of decreasing terms tending to zero is less than the first term omitted.

Clearly

$$\log_{n\to\infty} \frac{(\log n)^{r+2}}{n - \log n} = 0 .$$

Moreover, using the estimate $x! > (x/e)^x$

$$\lim_{n\to\infty} \frac{n^{r+2}}{(\log n)!} \leq \lim_{n\to\infty} \frac{n^{r+2}}{(\log n/e)^{\log n}}$$

$$= \lim_{n\to\infty} e^{(r+3)\log n - (\log n)^2}$$

$$= 0 .$$

Next

$$\lim_{n\to\infty} \frac{n^r}{(n+1)!}\left[\frac{1-\left(\frac{n+1}{n}\right)^{r+1}}{1-\left(\frac{n+1}{n}\right)}\right]$$

$$= \lim_{n\to\infty} \frac{(n+1)^{r+1} - n^{r+1}}{(n+1)!}$$

$$= 0 .$$

Finally

$$\lim_{n\to\infty} \frac{n^r}{(n-1)!} = 0$$

If we put

$$b_r = \sum_{k=0}^{\infty} \frac{k^r(-1)^k}{k!}$$

then we have just established the asymptotic expansion

$$\frac{F(n,2)}{n!} = \sum_{r=0}^{s} \frac{b_r}{n^r} + 0\left(\frac{1}{n^{s+1}}\right).$$

The numbers b_r can be computed as follows: First

$$b_0 = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} = \frac{1}{e}.$$

We now have

$$b_{r+1} = \sum_{k=0}^{\infty} \frac{k^{r+1}(-1)^k}{k!}$$

$$= \sum_{k=1}^{\infty} \frac{k^r(-1)^k}{(k-1)!}$$

$$= \sum_{k=0}^{\infty} \frac{(k+1)^r(-1)^{k+1}}{k!}$$

$$= -\sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \sum_{j=0}^{r} k^j \binom{k}{j}$$

$$= -\sum_{j=0}^{r} \binom{k}{j} b_j ,$$

or symbolically

$$b^{r+1} = -(1+b)^r .$$

This recurrence relation shows that b_r is always an integer multiple of $1/e$, say $b_r = (1/e)\, a_r$.

If we write

$$b(x) = \sum_{r=0}^{\infty} \frac{b_r}{r!} x^r ,$$

then similarly to the derivation of Eq. (3) we get

$$-b(x)\, e^x = \sum_{r=0}^{\infty} \frac{b_{r+1}}{r!} x^r = b'(x) .$$

Hence

$$b(x) = e^{-e^x} .$$

If c_r denotes the number of partitions of a set of r elements, then it is well known (Ref. 4) that

$$\sum_{r=0}^{\infty} \frac{c_r}{r!} x_r = e^{e^x - 1} .$$

Hence the a_r are the so-called Blissard or umbral inverses of the set partition function c_r (Ref. 3, p. 27).

The above results are summarized by the following theorem, which also gives the values of a_r for $0 \leq r \leq 20$.

Theorem 1. Let $F(n,2)$ be the number of permutations of $1, 2, \cdots, n$ with no runs of length 2 and put $G(n,2) = (1/n)\, F(n,2)$. Then

(i) $$n + (n-1)! = \sum_{k=0}^{n} \binom{n}{k} G(k,2) .$$

(ii) $$\sum_{k=0}^{\infty} \frac{G(k,2)}{k!} x^k = e^{-x}[1 - \log(1-x)] + x ,$$

$$\sum_{k=0}^{\infty} \frac{F(k,2)}{k!} x^k = xe^{-x}\left[\frac{x}{1-x} + \log(1-x)\right] + x + 1 .$$

(iii) $$F(n,2) = n! \sum_{k=0}^{n-1} \frac{(-1)^k}{k!} \frac{n}{n-k} + n(-1)^n, n \neq 1 .$$

(iv)
$$\frac{F(n,2)}{n!} \sim \frac{1}{e}\left(1 - \frac{1}{n^2} + \frac{1}{n^3} + \frac{1}{n^4} - \frac{2}{n^5} - \frac{9}{n^6} - \frac{9}{n^7} + \frac{50}{n^8} + \frac{267}{n^9} + \frac{413}{n^{10}} - \frac{2180}{n^{11}} - \frac{17731}{n^{12}} - \frac{50533}{n^{13}} + \frac{110176}{n^{14}} + \frac{1966797}{n^{15}} + \frac{9938669}{n^{16}} + \frac{8638718}{n^{17}} - \frac{278475061}{n^{18}} - \frac{2540956509}{n^{19}} - \frac{9816860358}{n^{20}} + \cdots + \frac{a_k}{n^k} + \cdots\right),$$

where

$$\sum_{k=0}^{\infty} \frac{a_k}{k!} x^k = e^{-e^x+1}.$$

In particular, the probability that a random permutation of $1, 2, \cdots, n$ has no runs of length 2 is

$$\frac{F(n,2)}{n!} \rightarrow \frac{1}{e} = 0.36788 \cdots \text{ as } n \rightarrow \infty.$$

3. An Asymptotic Formula for Arbitrary t

The main result of this section is the following theorem:

Theorem 2. Let $F(n,t)$ be the number of permutations of $1, 2, \cdots, n$ with no runs of length t. Then

$$\frac{F(n,3)}{n!} = 1 - \frac{1}{n} - \frac{3}{2}\frac{1}{n^2} - \frac{14}{3}\frac{1}{n^3} + 0\left(\frac{1}{n^4}\right),$$

$$\frac{F(n,4)}{n!} = 1 - \frac{1}{n^2} - \frac{5}{n^3} - \frac{29}{2}\frac{1}{n^4} + 0\left(\frac{1}{n^5}\right),$$

while for fixed $t > 4$

$$\frac{F(n,t)}{n!} = 1 - \frac{1}{n^{t-2}} - \frac{(t-2)(t+1)}{2}\frac{1}{n^{t-1}} - \frac{t(t+1)(3t^2-5t-10)}{24}\frac{1}{n^t} + 0\left(\frac{1}{n^{t+1}}\right).$$

Proof. The proof is based on the principle of inclusion and exclusion referred to earlier. Let $w(i_1, i_2, \cdots, i_r)$ be the number of permutations of $1, 2, \cdots, n$ with runs of length t beginning on the symbols $i_1, i_2, \cdots, i_r$. Let $W(r) = \sum w(i_1, i_2, \cdots, i_r)$, where the sum is taken over all subsets $\{i_1, i_2, \cdots, i_r\}$ of $\{1, 2, \cdots, n\}$ with r elements. It follows from the principle of inclusion and exclusion that

$$n! - W(1) + W(2) - W(3) < F(n,t) < n! - W(1) + W(2) - W(3) + W(4).$$

If we can show that

$$W(4)/n! = 0\left(\frac{1}{n^{t+1}}\right),$$

then

$$\frac{F(n,t)}{n!} = 1 - \frac{W(1)}{n!} + \frac{W(2)}{n!} - \frac{W(3)}{n!} + 0\left(\frac{1}{n^{t+1}}\right).$$

Then to complete the proof of Theorem 2 we need only calculate $W(1)$, $W(2)$, and $W(3)$.

We now show that for $t > 2$,

$$W(r)/n! = 0\left(\frac{1}{n^{t+r-3}}\right).$$

This will be seen to be false for $t = 2$, which explains why this case was handled separately. For each subset $T = \{i_1, i_2, \cdots, i_r\}$ of $S = \{1, 2, \cdots, n\}$, we associate a partition of n as follows: the set T partitions S into subsets of symbols which must remain adjacent in order for a permutation to have runs of length t beginning on $i_1, i_2, \cdots, i_r$. The number of elements in each subset is taken to be a term in the partition of n.

Example: Let $n = 15$, $t = 3$, $T = \{2, 5, 7, 8, 12\}$. Then S is partitioned into the subsets $\{1\}$, $\{2, 3, 4\}$, $\{5, 6, 7, 8, 9, 10\}$, $\{11\}$, $\{12, 13, 14\}$, $\{15\}$. This yields the partition $15 = 1 + 1 + 1 + 3 + 3 + 6$.

Let $n = b_1 + 2b_2 + \cdots + nb_n$ be the partition of n induced by T. Observe that $b_1 \geqq n - rt$, with equality holding when each element of T belongs to a distinct subset of the partition of S. Hence, there are at most $\binom{rt}{r}$ distinct partitions of S induced by all subsets T with r elements, since we can assume that r elements of T are chosen from a specified set of size rt.

ways of rearranging these subsets to give different permutations of S. Hence

$$W(r) \leq \binom{rt}{r} \max \frac{n^2 \left(\sum_{i=1}^{n} b_i - 1 \right)^2 !}{b_1!\, b_2! \cdots b_n!}, \tag{6}$$

where the maximum is taken over all partitions of n which can arise from subsets T of order r.

Now if $b_2 + \cdots + b_n = b$, then

$$\sum_{i=1}^{n} b_i \leq n - bt - (r - b) + b,$$

since $b_2 = b_3 = \cdots = b_{t-1} = 0$ and there are r runs of length t. Hence, from Eq. (6) we get

$$\frac{\cdots bt + 2b - r - 2) \cdots (n - bt + b - r)}{\cdots 2) \cdots (n - bt + 2b - r)}$$

Note that

$$\frac{F(n,t)}{n!} \geq 1 - \frac{W(1)}{n!} = 1 + 0\left(\frac{1}{n^{t-2}}\right),$$

so that this simple estimate suffices to show that the probability that a random permutation has no runs of length $t > 2$ approaches 1 as $n \rightarrow \infty$.

Two runs of length t can occur in one of two ways: (i) one run of length $t + 1, t + 2, \cdots, 2t - 1$, or (ii) two disjoint runs of length t. In the first case, one run of length $t + i$ can begin on any one of n symbols, leaving $(n - t - i)!$ ways of permuting the $n - t$ remaining symbols and n ways of shifting each permutation cyclically. In the second case, there are $n(n - 2t + 1)/2$ ways of choosing two disjoint runs of length t, $(n - 2t + 1)!$ ways of permuting the $n - 2t + 1$ subsets that remain when one run is fixed in place, and n ways of shifting each permutation cyclically.

Hence

$$W(2) = \sum_{i=1}^{t-1} n^2 (n-t-i)! + \frac{n^2(n-2t+1)}{2}(n-2t+1)!\,,$$

$$n \geqq 2t+1\,.$$

Three runs of length t can occur in one of three ways: (i) one run of length $t+2, t+3, \cdots, 3t-2$, (ii) one run of length t and one of length $t+1, t+2, \cdots, 2t-1$, or (iii) three disjoint runs of length t. In the first case, as before, one run of length $t+i$ can begin on any one of n symbols, leaving $(n-t-i)!$ ways of permuting the $n-t$ remaining symbols and n ways of shifting each permutation cyclically. Now, however, we get an additional factor of $i-1$, as there are $i-1$ places on which the middle run can begin. In the second case, there are $n(n-2t+1-i)$ ways of choosing two disjoint runs of length t and $t+i$, $(n-2t+1-i)!$ ways of permuting the $n-2t+1-i$ subsets that remain when one run is fixed in place, and n ways of shifting each permutation cyclically. In the third case, there are $n(n-3t+2)(n-3t+1)/6$ ways of choosing three disjoint runs of length t, $(n-3t+2)!$ ways of permuting the $n-3t+2$ subsets that remain when one run is fixed in place, and n ways of shifting each permutation cyclically.

Hence

$$W(3) = \sum_{i=2}^{2t-2} (i-1)\, n^2 (n-t-i)!$$

$$+ \sum_{i=1}^{t-1} n^2 (n-2t+1-i)(n-2t+1-i)!$$

$$+ \frac{n^2(n-3t+2)(n-3t+1)}{6}(n-3t+2)!\,,$$

$$n \geqq 3t+1\,.$$

We leave it to the intrepid reader to expand $1 - W(1)/n! + W(2)/n! - W(3)/n!$ in a power series in $1/n$ and verify that the terms given in the statement of the theorem are correct. With this the proof of Theorem 2 is complete.

It is evident that the above procedure can be continued to give the asymptotic expansion of $F(n,t)/n!$ to any desired accuracy.